NOTE ON A PROBLEM IN APPROXIMATION WITH AUXILIARY CONDITIONS*

BY DUNHAM JACKSON

Let $\rho(x)$ and $f(x)$ be two given functions of period $2\pi$, the former bounded and measurable, with a positive lower bound, the latter, for simplicity, continuous. Among all trigonometric sums $T_n(x)$, of given order $n$, there is one and just one for which the value of the integral

$$(1) \quad \int_0^{2\pi} \rho(x) [f(x) - T_n(x)]^2 dx$$

is a minimum. If the weight function $\rho(x)$ is identically 1; it is a matter of familiar knowledge that the minimum is reached when $T_n(x)$ is the partial sum of the Fourier series for $f(x)$. A considerable amount of attention has been given recently to the problem of the convergence of the minimizing sum $T_n(x)$ toward $f(x)$, as $n$ becomes infinite, under the generalized conditions that result from the admission of an arbitrary weight function.†

Let $x_1, \cdots, x_N$ be $N$ values of $x$ in the interval $0 \leq x < 2\pi$. The problems of the preceding paragraph may be further varied by admitting to consideration only such sums $T_n(x)$ as satisfy the conditions

$$(2) \quad T_n(x_i) = f(x_i), \quad (i = 1, 2, \cdots, N),$$

and inquiring after the minimum of the integral (1) subject to these auxiliary conditions. It is understood that the given value of $n$ is large enough so that the conditions (2) can be

---

* Presented to the Society, April 3, 1926.
fulfilled; for this it is sufficient that \( n \geq \frac{1}{2}(N - 1) \). There is no essential novelty in the proof of the existence and uniqueness of the sum which yields the minimum. The notation \( T_n(x) \) being restricted henceforth to this "approximating sum," it is the purpose of the following lines to discuss the convergence of \( T_n(x) \) toward \( f(x) \), as \( n \) increases without limit. The question is not trivial, even if \( \rho(x) = 1 \). It is quite distinct from the conventional problems of interpolation, inasmuch as \( N \) is fixed, and does not increase with \( n \).

For each value of \( n \left( \geq \frac{1}{2}(N - 1) \right) \), let a continuous function \( \varphi_n(x) \) be defined, having \( \epsilon_n \) as an upper bound for its absolute value, and such that

\[
\varphi_n(x_i) = 0, \quad (i = 1, 2, \ldots, N).
\]

Let \( \tau_n(x) \) be the approximating sum of the \( n \)th order for \( \varphi_n(x) \); that is, the sum which minimizes the integral

\[
\int_0^{2\pi} \rho(x) \left[ \varphi_n(x) - \tau_n(x) \right]^2 dx,
\]

subject to the conditions

\[
\tau_n(x_i) = 0, \quad (i = 1, 2, \ldots, N).
\]

Exactly as in the absence of the restrictions (3), (4), it may be shown* that

\[
|\varphi_n(x) - \tau_n(x)| \leq k\epsilon_n\sqrt{n},
\]

where \( k \) is independent of \( n \) (expressible, in fact, in terms of the ratio of the upper and lower bounds of \( \rho(x) \), and independent of anything else). The new conditions call for notice only to the extent of the observation that a trigonometric sum which vanishes identically comes within the requirements of (4). Furthermore, it is recognized at once that if \( \varphi_n(x) \) is defined by the relation

\[
\varphi_n(x) = f(x) - t_n(x),
\]

where \( t_n(x) \) is a trigonometric sum of the \( n \)th order taking on the same values as \( f(x) \) at the points \( x_1, x_2, \ldots, x_N \), then \( \tau_n(x) \) and \( T_n(x) \) are related by the identity†

---

* D. Jackson, this BULLETIN, loc. cit.
† Cf. this BULLETIN, loc. cit., p. 261.
\[ \tau_n(x) = T_n(x) - t_n(x) , \]
so that \( f(x) - T_n(x) = \varphi_n(x) - \tau_n(x) \) identically, and
\[ |f(x) - T_n(x)| \leq k\epsilon_n \sqrt{n} . \]

The formulation of sufficient conditions for convergence is reduced then to the determination of the order of magnitude of \( \epsilon_n \), the measure of the accuracy with which \( f(x) \) can be uniformly approached by trigonometric sums \( t_n(x) \) such that
\[ (5) \quad t_n(x_i) = f(x_i) , \quad (i = 1, 2, \ldots, N). \]

Let \( y_1, y_2, \ldots, y_N \) be any \( N \) numbers subject to the conditions \( |y_i| \leq 1, i = 1, 2, \ldots, N \). If \( N \) is even, let \( x_0 \) be a point in \( (0, 2\pi) \) distinct from \( x_1, x_2, \ldots, x_N \), and let \( y_0 = 1 \). Let \( t(x) \) be the trigonometric sum of order \( \frac{1}{2}(N-1) \) or \( \frac{1}{2}N \), according as \( N \) is odd or even, which takes on the values \( [y_0], y_1, \ldots, y_N \) at the points \( [x_0], x_1, \ldots, x_N \). Let \( g \) be the maximum of \( |t(x)| \). This \( g \) is a continuous function* of \( y_1, y_2, \ldots, y_N \), and has a maximum \( G \), as the \( y \)'s range over all admissible values. If 1 is replaced by \( \eta \) as upper bound for the absolute values of the \( y \)'s, the greatest possible absolute value of the corresponding \( t(x) \) is \( G\eta \).

Now suppose it is known that for each \( n \geq \frac{1}{2}N \) there is a trigonometric sum \( \tilde{t}_n(x) \), of the \( n \)th order, satisfying everywhere the relation
\[ |f(x) - \tilde{t}_n(x)| \leq \eta_n , \]
but not further specially restricted at the points \( x_1, \ldots, x_N \). Let
\[ y_i = f(x_i) - \tilde{t}_n(x_i) , \quad (i = 1, 2, \ldots, N) , \]
and let \( t(x) \) be determined as above, for this set of \( y \)'s. Then \( |t(x)| \leq G\eta_n \), where \( G \) is independent of \( n \) (being dependent only on \( x_1, \ldots, x_N \)). The determination of

* Explicitly, as is well known,
\[ t(x) = \sum_{i=1}^{N} y_i \sin \frac{1}{2}(x-x_1) \cdots \sin \frac{1}{2}(x-x_{i-1})\sin \frac{1}{2}(x-x_{i+1}) \cdots \sin \frac{1}{2}(x-x_N) \]
when \( N \) is odd, the initial index 1 being replaced by 0 when \( N \) is even.
$t(x)$ is different for different values of $n$, but each $t(x)$ is itself a trigonometric sum of order $\frac{1}{2}N$ at most. The identity

$$t_n(x) = \tilde{t}_n(x) + t(x)$$

defines a sum of the $n$th order such that the conditions (5) are fulfilled, and such that

$$|t(x) - t_n(x)| \leq (1 + G)\eta_n.$$  

As the factor $1 + G$ is independent of $n$, this means that the order of the attainable approximation is not affected by the imposition of the restrictions (5).

In particular, if $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leq \delta$, and if $\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0$, sums $\tilde{t}_n(x)$ will exist such that $\lim_{n \to \infty} \eta_n \sqrt{n} = 0$, and there will consequently be sums $t_n(x)$ such that $\lim_{n \to \infty} \epsilon_n \sqrt{n} = 0$. We may state the result as a theorem, identical in form with the one found when the auxiliary conditions (2) are omitted.

**Theorem.** The sum $T_n(x)$ will converge uniformly to the value $f(x)$ for $n \to \infty$, provided that

$$\lim_{\delta \to 0} \frac{\omega(\delta)}{\sqrt{\delta}} = 0.$$  

It is readily seen that essentially the same treatment can be carried through if $[f(x) - T_n(x)]^2$ in (1) is replaced by $|f(x) - T_n(x)|^m$, for any value of $m > 1$; the condition for convergence is that $\lim_{\delta \to 0} \omega(\delta)/\delta^{1/m} = 0$. The discussion can be further extended in various ways that need not be elaborated here.

**The University of Minnesota**

* Cf. this Bulletin, loc. cit., p. 261.*