TONELLI ON CALCULUS OF VARIATIONS

Fondamenti di Calcolo delle Variazioni. By Leonida Tonelli. Bologna, Zanichelli. Vol. 1, 7+406 pp., 1921, 55 Lire; vol. 2, 8+660 pp., 1923, 80 Lire.

In its classical period, the calculus of variations depended for many of its pivotal theorems upon the theory of differential equations. This debt was partly offset by the contributions to the theory of boundary value problems made by Mason, Richardson, et al., in which existence and oscillation theorems were obtained, via the theory of integral equations, from the calculus of variations. But it is an entirely different procedure which underlies Tonelli's work. For the line integral whose extreme values are sought, is here studied by means of its direct functional dependence upon the curve along which it is taken. This is a new departure, at least in so far as its systematic use throughout the theory is concerned. Nothing of the sort is found in the texts based upon the Weierstrass theory (Kneser, Bolza). And there is not more than a suggestion of it in the first volume of Hadamard's *Leçons*.

By putting into the foreground the dependence of the integral upon the entire curve, the author focuses attention upon the "integral" aspects of the theory, rather than upon its "differential" characters. We get a calculus of variations which is concerned primarily with neighborhoods of curves, rather than with those of points. Another striking feature of the new theory, and one intimately connected with what has already been mentioned, is the preponderant importance of the theory of absolute extrema, which now assumes first place while the theory of relative extrema, which heretofore claimed the lion's share, is derived from it. There is possible, of course, a difference of opinion with regard to the fundamental significance of this new theory. Whether independence from the theory of differential equations is a desideratum may well be queried. It is very likely true that, for a long time to come at least, the classical theory will supply the most ready method of becoming acquainted with the calculus of variations and its problems. It must be admitted also that the importance of Tonelli's method would be greatly enchanced if it should prove possible to extend his theory to more general functionals than those which occur in the calculus of variations. On the other hand, the advances in the theory which the treatment of the integral as a function of lines makes possible seem to secure a permanent place for it.

The task of the reviewer has been considerably lightened by an article contributed by the author to volume 31 of this BULLETIN (p. 163), in which he explained how the functional conception of the calculus of varaitions led to the use of semi-continuity and related ideas which play a role of central importance in the theory. Volume I, for which its author was awarded the 1923 gold medal of the Italian Society of Sciences, is devoted to a very careful and complete development of such preliminary material as his mode of treatment of the calculus of variations makes necessary. It contains, besides a historical sketch of the development of the subject, Parts I, II, III of 200, 145, and 120 pages respectively.

The first of these gives a useful treatment of some important concepts and methods which have come into use in the theory of functions of a real variable during recent years. Among them is a discussion of the Lebesgue integral which, in order to avoid implied acceptance of the Zermelo axiom, substitutes the concept of "pseudointerval" for that of "measurable set," and which follows closely the treatments of W. H. Young and de la Vallée-Poussin. In this connection we call attention to the further development of his ideas concerning the Lebesgue integral which Tonelli has given in a very interesting article in the ANNALI DI MATEMATICA, ser. 4, vol. 1 (1924). One finds here a sequence of theorems of Ascoli, Arzelà, Hilbert and Tonelli giving conditions which insure that an infinite set of functions or of curves possess a limiting element; the concepts of the "equivalence of a point set to an interval," of absolute continuity, of quasi-continuity, and the well known theorems on differentiability and other properties of the Lebesgue indefinite integral. The treatment excells in clarity of exposition and elegance of treatment, combining a wholesome regard for necessary details with emphasis upon essentials. It collects a great deal of material that is scattered in various journals and of which a good share is due to the author, presenting it in such a manner as to make it available to the student who has had a first course in the real variable.

After this introductory section we enter upon an alternating sequence maintained throughout both volumes, in which treatment of a certain problem in the parametric form is succeeded by the discussion of the same problem in the "ordinary" form (usually called the x-form). Part II entitled "Functions of Lines—Parametric Form" discusses the integral $\mathfrak{J}_C = \int_0^L F(x, y, dx/ds, dy/ds)ds$ as a function of the curve C defined by by $x=x(s)$, $y=y(s)$, $0 \leqq s \leqq L$. The curves C admitted are "ordinary curves," i.e., rectifiable continuous curves which lie entirely in a certain domain A of the XY-plane; the function $F(x,y,x',y')$ is subjected to conditions of continuity, differentiability, and to the homogeneity condition. Of great convenience for the sequel is the classification of integrals $\mathfrak{J}_C$ in accordance with the sets for which the functions F and F_1 vanish. This renders possible a systematic treatment of various cases which may arise and which heretofore have not always been clearly recognized. Thus, besides the regular and definite cases in which F_1 and F respectively are different from zero for all points (x,y) of A and for all points (x',y') for which $x'^2+y'^2 \neq 0$, there are considered the cases of quasi-regular, normally-quasi-regular, semi-normally-quasi-regular, and semi-definite problems characterized by restrictions on the loci of the zeros of the functions F_1 and F.

With these preparations out of the way, the author is now ready to attack the main problem of his first volume, viz., the determination of necessary and sufficient conditions for the semi-continuity of $\mathfrak{J}_C$. That continuity (defined in terms of "a neighborhood of a curve," as developed in

Part I), is too strong a condition to impose upon $\mathfrak{J}_C$ if the theory is to be at all applicable is made evident by the remark that the length integral $\int\sqrt{x'^2+y'^2}dt$ does not satisfy it, because it lacks upper semi-continuity; it is lower semi-continuous. In the next two chapters we find necessary and sufficient conditions for the semi-continuity of the function $\mathfrak{J}_C$ over the entire region A, or on a particular curve C. Much in these chapters consists of the author's own researches; there is in them a great deal to delight the reader who is sensitive to the beauties of a mathematical proof. Let me call attention to the proof that the condition $F_1(x,y,x',y')\geqq 0$ for every normalized pair (x',y'), throughout the set of points which are either interior to A or are limiting points of such points, is necessary for lower semi-continuity of $\mathfrak{J}_C$; to the theorem that continuity of $\mathfrak{J}_C$ requires that $F\equiv P(x,y)x'+Q(x,y)y'$ at every point of the set just mentioned; to the theorem that if $\mathfrak{J}_C$ is positively definite and the region A is of a certain character, then it is both necessary and sufficient for the lower semi-continuity of $\mathfrak{J}_C$ on an ordinary curve C_0 that the inequality $E(x_0,y_0;\cos\theta_0,\sin\theta_0;\cos\theta,\sin\theta)\geqq 0$ hold nearly everywhere on C_0. This last result, giving refreshingly novel significance to the Weierstrass condition, is of particular interest to anyone who knows what an important role this condition plays in the classical theory. Chapter 8, concerned with further properties of the integral $\mathfrak{J}_C$, begins with the theorem that if C_0 is an ordinary curve of length L_0 then every $e>0$ determines a neighborhood of C_0 and a number $d>0$ such that for every curve in this neighborhood, the inequality $|L-L_0|<d$ carries with it that $|\mathfrak{J}_C-\mathfrak{J}_{C_0}|<e$; this is followed by a second theorem to the effect that the second of the inequalities just mentioned implies the first, at least if $\mathfrak{J}_C$ is regular. Of this latter result a number of extensions are developed relaxing the conditions on $\mathfrak{J}_C$ and replacing L by a general integral, running in close analogy to the theorems of Lindeberg. The remainder of the chapter is taken up by convergence and comparison theorems, of which the following is a sample: If $\mathfrak{J}_C$ is normally quasi-regular and if the sequence of ordinary curves $\{C_n\}$ converges uniformly to the ordinary curve C_0, then $\{\mathfrak{J}_{C_0}\}\rightarrow\mathfrak{J}_{C_0}$ carries with it that

$$\left\{\int_{C_n} Gds\right\}\rightarrow\int_{C_0} Gds,$$

if G is positively homogeneous of degree 1 in x' and y', and continuous in A and for every nonsingular pair (x',y').

At the end of this chapter the reader has reached the climax of Volume I. The stage is set for the developments of the second volume. The only thing left is to treat the integral $I=\int f(x,y,y')dx$ as the integral $\mathfrak{J}_C$ was treated before; and this is done in Part III. While the results obtained here are in most instances entirely parallel to those of Part II, a considerable number of important changes are necessary in the proofs.

In Volume II, the calculus of variations proper begins. Part I (pp. 1–278) deals with the parametric case. The first question treated is that of the existence of absolute extremands among the elements of classes K of ordinary curves C, i.e. the existence of ordinary curves C of K such that for every curve of K we shall have $\mathfrak{J}_{C_0}\geqq\mathfrak{J}_C$ or $\mathfrak{J}_{C_0}\leqq\mathfrak{J}_C$. Two interesting general criteria are followed by an array of twelve existence theorems for the integral in parametric form, which cover a wide group of cases, including all those which arise in the classical problems and applications, and in which semi-continuity plays an important role. The classes K of curves relative to which the "absolute" extremand (the writer is aware here of a confrontation of opposites) is considered, are "complete classes" of ordinary curves, i.e. such as contain their limiting curves whenever these are rectifiable. The first of the existence theorems is a generalization of Hilbert's well known theorem; the subsequent theorems apply to integrals upon which various less restrictive conditions are imposed. Next follow extensions to the case in which the region A is unlimited; and the existence of "extremands 'in piccolo' " closes the first chapter. In Chapter II, the author proceeds to discuss the properties of extremands. And it is here that we get for the first time a more intimate linking up with the classical theories. The cases of fixed end-points and of one or two variable end-points are treated in close parallelism. It is shown that the conditions of Legendre and of Weierstrass are necessary for an extremand C_0 and that along every arc of C_0 which possesses a continuously turning tangent, the Euler equations must be satisfied; an ordinary curve which has this property and which satisfies the Euler equations is called an *extremal*. It is shown that every ordinary minimizing curve must satisfy the equations

$$\int_0^s F_x ds - \frac{d}{ds}\int_0^s F_{x'}ds = C_1, \quad \int_0^s F_y ds - \frac{d}{ds}\int_0^s F_{y'}ds = C_2;$$

an ordinary curve satisfying these equations is called an *extremaloid*. The corner point condition, the transversality condition, the envelope theorems are derived in about the usual way. A much more detailed treatment than has hitherto been available is given of the conditions on the boundary of the domain A.

In Chapter III various existence theorems for extremals, their dependence upon the coordinates of the end-points, the continuity and differentiability of the functions representing them, all of which are usually obtained in consequence of extensions of theorems on differential equations are here proved directly by a method based upon the foundations laid in Chapter II concerning the existence of extremands and their relation to extremals. The last part of this chapter deals with broken extremals, here called "simple extremaloids," along the lines followed in the work of Carathéodory.

The following chapter is concerned with the theorems of Darboux and Osgood, and with Jacobi's condition. The latter is treated on the basis of a definition of "the focus of the point P on the extremal E_0," which may be approximately rendered thus: the point P_0 is called the right-hand focus of P on E_0, if it is the first point following P, such that for every ρ there exists on E_0 or on its extension a point P^* belonging to an extremal $E(P,P^*)$, not containing nor contained in the arc $E_0(P,P_0)$ and lying in a second-order ρ-neighborhood of $E_0(P,P_0)$. In this statement I have not specified what is meant by a "second-order ρ-neighborhood of a curve," because the general character of the meaning of this phrase is doubtless sufficiently clear from its use in other places, and because the statement as given should suffice to indicate the characteristics of the author's mode of approach. The definition is followed by that of right and left-hand focus of a curve and then by the analytic determination of foci. The use of the word "focus" in such a way as to cover not merely the case of end-points variable along a curve but also that of fixed end-points, is becoming more and more general and has much to recommend it. The different cases, leading to the conditions of Jacobi, Kneser and Bliss, are treated in close proximity. They are followed by the condition of Poincaré for closed extremals and that of Carathéodory (as developed by Bolza) for broken extremals. There is also given, in small print, a derivation of Jacobi's condition by the classical method, based upon the properties of second-order linear differential equations and the transformation of the second variation. It is to be regretted that in the discussion of the second variation no mention is made of the elegant method which Bliss has used in recent years to obtain the Jacobi condition and the key idea of which lies in associating a new minimizing problem with the condition that the second variation be positive (see this BULLETIN, vol. 26 (1920), p. 343). The chapter ends with theorems concerning the uniqueness of the extremal joining two points.

Passing over the first four chapters of Part II (pp. 281–461) in which the ordinary case is treated we turn to Chapter IX, with which Part II closes and which takes up a number of the classical problems. One wonders upon approaching this chapter what effect the point of view of Tonelli is going to have upon the actual solution of problems, whether after all, when it comes to minimizing a particular definite integral, the old methods will not have to be used. To some extent this is indeed the case; there are, however, some interesting modifications in the manner of treatment. Take, for instance, the treatment of the classical problem of finding the surface of revolution of minimum area. The first fact established is that, in view of the general existence theorems, there always exists at least one minimizing curve P_1P_2 and that every arc of this curve which has no point in common with the axis of revolution (taken to be the X-axis) must be an extremal. It is then shown that if the minimizing curve has no points on the X-axis, it is an arc of catenary, if P_1 and P_2 have different abscissas, and a line parallel to the Y-axis if these abscissas are equal; also that, if the minimizing curve has at least one point on the X-axis, it consists of the broken line $P_1Q_1Q_2P_2$ formed by the perpendiculars from P_1 and P_2 to the X-axis, together with the segment of the X-axis between the projections Q_1 and Q_2 of these points. It follows that the solution is unique if P_1 and P_2 have the same abscissas and also if P_1 or P_2 are on the X-axis. For the case in which P_1 and P_2 are both above the X-axis and have distinct abscissas, it is shown that, if P_2 lies on a certain curve S, there are two curves furnishing an absolute minimum, one being an arc of catenary and the other the broken line $P_1Q_1Q_2P_2$. If P_2 lies in the one or the other of the two regions determined by S, there is a unique minimizing curve, in the one case the arc of catenary, in the other case the broken line. It is surprising that in the presentation of these results which are well known, although the author's method of obtaining them has some elements of novelty, no mention is made of the work of MacNeish and Sinclair, to whom doubtless they are largely due. A similar omission is to be recorded with regard to Mason's work on the problem of the minimum moment of inertia. But to return to the treatment of the examples: While we shall still be obliged to solve differential equations for the discovery of extremals, the new method furnishes in a more direct way than was possible before, complete information concerning the existence of absolute extrema and the curves by which these are attained.

The third part of Volume II (pp. 465–567) deals with the isoperimetric problems, first in parametric form and then in the ordinary form. The method of treatment followed here is entirely analogous to that used in the earlier parts of the book. The existence theorems again take care of a very broad class of problems.

The fourth and final part (pp. 571–639) is concerned with the problem that usually receives first consideration, viz., that of the relative minimum. We find here the distinction between weak and strong extrema, sets of necessary and sufficient conditions for each of these, the introduction of the field concept, etc. Both the unconditioned problem and the isoperimetric problem are taken up in the parametric and in the ordinary forms, for fixed and for variable endpoints. That this can be done in so few pages is, of course, due to the fact that the necessary and sufficient conditions for the existence of absolute extrema can be turned to use by appropriate specification of the class K of curves to which these conditions refer.

Little has to be added to this review. How useful the new method of treatment which Tonelli has introduced into the calculus of variations is going to be, will depend upon many things. Its value as a means of establishing the existence of absolute extrema for the cases thus far treated must be recognized; as a means of actually discovering extremizing curves, it has to rest upon the classical theory, which thus it extends rather than displaces. How far the theory can be extended so as to take care of the more complicated problems of the calculus of variations and so as to furnish a theory of maxima and minima in the functional calculus, only the future can tell. I do not at present see how it could be used for the purpose of initiating students into the subject; but, those who have been made familiar with the outstanding facts of the theory by the classical methods, and who have the proper acquaintance with the spirit and method of the real function theory should be able to use these books to great advantage and to find in them an ample field for study and research. I have already spoken of the clearness and elegance with which they are written. To this I only wish to add that the typography is excellent and that the very complete indexes and the systematic use of cross-references make it very convenient to read them. I am looking forward with great interest to the further volumes on this subject which the author has promised for the near future.

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