GENERALIZATION OF LAGRANGE'S THEOREM

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1. Introduction. The following theorem due to Lagrange is of considerable importance in the theory of equations.

LAGRANGE'S Theorem. If the group to which the rational function \( \psi(x_1, \cdots, x_n) \) belongs is a subgroup of the group to which the rational function \( \phi(x_1, \cdots, x_n) \) belongs, then \( \phi \) equals a rational function of \( \psi \) and the elementary symmetric functions of the variables \( x_1, \cdots, x_n \).

In this paper I prove a similar theorem for sets of variables.

2. Notation and Definitions. Consider the \( n \) sets of \( m \) variables \( x_{1i}, x_{2i}, \cdots, x_{mi} \) (\( i = 1, \cdots, n \)), which may be regarded as coordinates of \( n \) points in \( m \)-space. By a permutation of these sets of variables we mean a permutation of the points. Thus a permutation which changes \( x_{1i} \) to \( x_{lj} \), also changes \( x_{2i}, \cdots, x_{mi} \) to \( x_{2j}, \cdots, x_{mj} \) respectively. It is simpler to regard the permutation as affecting the second subscripts of the variables, with the above notation, than as affecting the \( x \)'s.

A function \( \phi(x_{11}, x_{21}, \cdots, x_{1m}; \cdots; x_{1n}, x_{2n}, \cdots, x_{mn}) \) is said to belong to a substitution group \( G \) on the symbols \( 1, 2, \cdots, n \), if \( \phi \) is unaltered by every substitution of \( G \) and by no substitution on these symbols not contained in \( G \). There exist functions which belong to a given substitution group. In fact, we can construct such functions involving only the variables \( x_{11}, x_{12}, \cdots, x_{1n} \).*

3. A Generalization. We proceed to prove the following generalization of Lagrange's Theorem.

* Netto, Substitutionentheorie und ihre Anwendung auf die Algebra, 1882, p. 27.
THEOREM. If the group to which the rational function
\[ \psi(x_{11}, x_{21}, \ldots, x_{m1}; \ldots; x_{1n}, x_{2n}, \ldots, x_{mn}) \]
belongs, is a subgroup of the group to which the rational function
\[ \phi(x_{11}, x_{21}, \ldots, x_{m1}; \ldots; x_{1n}, x_{2n}, \ldots, x_{mn}) \]
belongs, then \( \phi \) equals a rational function of \( \psi \) and the elementary symmetric functions of the sets of variables \( x_{1i}, x_{2i}, \ldots, x_{mi}, (i=1, \ldots, n) \).

It will suffice to consider the case \( m=3 \). The elementary symmetric functions of the \( n \) triads of variables are defined by*

\[ p_{ijk} = \sum x_{11} x_{12} \cdots x_{1i} x_{21} x_{2i} x_{31} \cdots x_{2i} x_{32} x_{3i} + \cdots \cdot x_{2i} x_{2i} x_{3i} \cdot \cdot \cdot x_{3i} x_{3i} \cdot \cdot \cdot x_{3i} \cdot \cdot \cdot x_{3i} \cdot (i + j + k \leq n). \]

With the aid of these functions, we can express any one of the variables \( x_{1i}, x_{2i}, x_{3i} \) as a rational function of any one of the others. In fact,† we have

\[
\begin{align*}
x_{1i} &= \frac{\frac{p_{100} x_{3i}^{n-1} - p_{101} x_{3i}^{n-2} + p_{102} x_{3i}^{n-3} - \cdots}{nx_{3i}^{n-1} - (n-1)p_{001} x_{3i}^{n-2} + (n-2)p_{002} x_{3i}^{n-3} - \cdots}}, \\
x_{2i} &= \frac{\frac{p_{010} x_{3i}^{n-1} - p_{011} x_{3i}^{n-2} + p_{012} x_{3i}^{n-3} - \cdots}{nx_{3i}^{n-1} - (n-1)p_{001} x_{3i}^{n-2} + (n-2)p_{002} x_{3i}^{n-3} - \cdots}}.
\end{align*}
\]

Hence every function of the triads of variables can be expressed as a function of \( x_{31}, x_{32}, \ldots, x_{3n} \), with coefficients that belong to the symmetric group. In particular, suppose

\[
\psi(x_{11}, x_{21}, x_{31}; \ldots; x_{1n}, x_{2n}, x_{3n}) = \psi_1(x_{31}, x_{32}, \ldots, x_{3n}),
\]

\[
\phi(x_{11}, x_{21}, x_{31}; \ldots; x_{1n}, x_{2n}, x_{3n}) = \phi_1(x_{31}, x_{32}, \ldots, x_{3n}).
\]

Evidently \( \psi \) and \( \psi_1 \) belong to the same group \( H \), and \( \phi \) and \( \phi_1 \) belong to the same group \( G \). As \( H \) is a subgroup of \( G \) by hypothesis, it follows from Lagrange’s Theorem, that \( \phi \) equals a rational function of \( \psi \) and the elementary symmetric functions \( p_{001}, p_{002}, \ldots, p_{00n} \). The theorem follows.

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* See Bôcher, Higher Algebra, p. 252.