ON THE CONVERGENCE OF TRIGONOMETRIC APPROXIMATIONS FOR A FUNCTION OF TWO VARIABLES*

BY ELIZABETH CARLSON

A discussion of the convergence of approximating functions for a given function of two variables $f(x, y)$, just as in the case of functions of one variable, can be based on two sets of theorems: (1) theorems on the existence of functions of closest approximation; (2) theorems on the representation of $f(x, y)$ by means of finite sums constructed in a specific way.

From the first group we shall make use of the following theorem:

**Theorem I.** Let $p_1(x, y), p_2(x, y), \ldots, p_N(x, y)$ be $N$ functions of $x$ and $y$, continuous in the region $R$: $(a \leq x \leq b, c \leq y \leq d)$, and linearly independent in this region. Let

$$
\phi(x, y) = c_1p_1(x, y) + c_2p_2(x, y) + \cdots + c_Np_N(x, y)
$$

be an arbitrary linear combination of the given functions with constant coefficients. Let $f(x, y)$ be a continuous function of $x$ and $y$ in $R$. Then there exists a choice of the coefficients $c_k$ in $\phi(x, y)$ such that the integral

$$
\int_a^b \int_c^d |f(x, y) - \phi(x, y)|^m dy, \quad (m > 0),
$$

has its minimum value. The function $\phi(x, y)$ so determined is unique for $m > 1$. It is called an approximating function for $f(x, y)$ corresponding to the exponent $m$.

This theorem can be proved by methods analogous to those used in proving the corresponding theorem for functions of a single variable.† In this paper we shall choose

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the $p$'s so that a function of the form $\phi(x, y)$ with arbitrary coefficients is an arbitrary trigonometric sum of order $n$ in $x$ and of the same order in $y$, and we shall take as the region $R$ the square $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$.

From the second group of theorems, we shall apply the following theorems.

**Theorem II.** If $f(x, y)$, of period $2\pi$ in each argument, has a modulus of continuity $\omega(\delta)$, then $f(x, y)$ can be represented everywhere by a trigonometric sum $t_n(x, y)$, of order $n$ in each variable, with an error that does not exceed a constant times $\omega(2\pi/n)$.

**Theorem III.** If $f(x, y)$ has continuous first partial derivatives, it can be approximately represented with a maximum error $\varepsilon_n$ such that $\lim_{n \to \infty} n\varepsilon_n = 0$.

The assertion II can be proved once more by an adaptation of the reasoning used in the case of functions of a single variable.* A proof of III was communicated to the writer orally by Professor Jackson.

Theorems I and II lead to the following theorem for the convergence of $T_n(x, y)$, the trigonometric approximating function, of order $n$ in each variable, corresponding to the exponent $m$ ($m > 1$).

**Theorem IV.** If $f(x, y)$, of period $2\pi$ in each argument, is everywhere continuous, with a modulus of continuity $\omega(\delta)$, then a sufficient condition for the uniform convergence of $T_n(x, y)$ to $f(x, y)$ is†

$$\lim_{\delta \to 0} \frac{\omega(\delta)}{\delta^{2/m}} = 0.$$
The occurrence of the exponent $2/m$, where $1/m$ is found in the case of functions of a single variable, is due to the fact that the magnitude of an interval of length $1/n$ is replaced in the course of the present reasoning by the area of a square having a quantity of the order of $1/n$ for the length of its side.

This result is significant only if $m > 2$, since otherwise the condition as stated requires that $f(x, y)$ be constant. When $m = 2$, it is sufficient (in consequence of III) that $f(x, y)$ have continuous first partial derivatives. There are corresponding results for $m < 2$.

The reasoning is not materially changed if the $m$th power in the integral to be minimized is multiplied by a positive measurable weight function $\rho(x, y)$ with a positive lower bound.

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ON A TENSOR OF THE SECOND RANK IN FUNCTION SPACE*

BY DUNHAM JACKSON

In a recent paper† the writer has discussed a doubly infinite matrix of derivatives which has the properties of a tensor of the second rank in function space, and a quantity obtained by contraction of this tensor, which is analogous to a divergence. The latter concept is suggested formally by the writing of an infinite series, the general term of which in many cases does not approach zero; but an example was constructed in which the series is convergent, and defines a quantity which can be alternatively expressed in a form independent of any particular coordinate system.

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