ON THE INTEGRO-DIFFERENTIAL EQUATION OF
THE BÔCHER TYPE IN THREE-SPACE

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1. Introduction. Bôcher has shown* that if a function
\( f(x, y) \) is continuous and has continuous first partial derivatives
in a region \( R \) and satisfies the condition
\[
\int_C \frac{\partial f}{\partial n} \, ds = 0
\]
for every circle \( C \) lying entirely in \( R \), then \( f(x, y) \) is harmonic
at each interior point of \( R \). Bôcher treats only functions
in two variables and by a method which cannot be directly
extended to three-space.

It is the purpose of the present note to show, by a simple
modification of the second part of Bôcher's argument,
that this result may at once be extended to three-space, and
also to investigate the nature of the function \( f \) if Bôcher's
condition of continuity is somewhat weakened. We shall
treat explicitly functions in three variables only, but it will
easily be seen that with a slight modification the statements
of Theorem II are applicable to two-space as well.

**Theorem I.** If a function \( f(x, y, z) \) is continuous, and
has continuous first partial derivatives in a connected finite
region \( R \), and is such that the surface integral \( \int_S (\partial f/\partial n)ds \)
vanishes when taken over every sphere \( S \) lying in \( R \), then at
each interior point of \( R \), \( f \) is harmonic; that is, it satisfies
Laplace's equation
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0
\]
at each interior point of \( R \).

In the preceding integral, as well as in what follows, the derivative $\frac{df}{dn}$ is to be taken either toward the interior of $S$, or toward the exterior of $S$, throughout the region of integration. Let $P$ be any interior point of $R$ and consider two spheres $S_1$ and $S_2$ of radii $r_1$ and $r_2 < r_1$ with centers at $P$. By hypothesis, we have

$$\int_S \frac{df}{dn} \, ds = 0;$$

or, setting $ds = r^2 \, d\omega$, where $d\omega$ is the element of area on the unit sphere with center $P$,

$$\int_S \frac{df}{dn} \, d\omega = 0.$$

It follows that

$$\int_{r_2}^{r_1} \, dr \int_S \frac{df}{dn} \, d\omega = 0.$$

Because of the continuity of $f$ and its derivatives the order of integrations in the above integral may be inverted and we have

$$\int_{S_1} f \, d\omega - \int_{S_2} f \, d\omega = 0.$$

Let $f(P)$ be the value of $f$ at the point $P$. Then since $f$ is continuous at $P$ we obtain from (2) by letting $r_2$ approach zero,

$$f(P) = \frac{1}{4\pi} \int_{S_1} f \, d\omega = \frac{1}{4\pi r_1^2} \int_{S_1} f \, ds.$$

We thus see that our function $f$ possesses the so-called mean-value property, that is, its value at the center of any sphere is the mean of its values on the surface of the sphere.

Consider now the function $F$ which takes the same values as $f$ on $S_1$ and which is harmonic interior to $S_1$. This function exists and can be expressed as a Poisson integral. It is well known that $F$ also possesses the mean-value prop-
ergy and hence so also does the difference \( f - F \). But a continuous function having the mean value property in a closed region \( R \) must take its greatest and least values on the boundary of \( R \). Since the difference \( f - F \) is identically zero on \( S_1 \) it follows that it is zero everywhere within \( S_1 \) and hence \( f \) must be harmonic at \( P \) as was to be proved.

It is evident that the original hypothesis that

\[
\int \frac{\partial f}{\partial n} ds = 0
\]

about every sphere in \( R \) is unnecessarily broad. All that is needed in the above proof is that each point \( P \) may be surrounded by a region, no matter how small, which is such that the above integral vanishes when taken over every sphere lying entirely within it.

2. A More General Theorem. We shall now weaken the original condition of continuity on \( f \) and suppose that it is continuous at every interior point of \( R \) except possibly at a finite number of points \( P_1, P_2, \ldots, P_n \). We shall refer to these exceptional points in the sequel as the points \( P_i \). Our other condition on \( f \) now takes the form "about each interior point of \( R \) there exists a region \( M \) which is such that in its interior \( f (\partial f/\partial n)ds \) evaluated over every sphere which lies in \( M \) and does not pass through one of the \( P_i \) is zero."

It is sufficient that if \( M \) contains one of the exceptional points, it contains only one.

That \( f \) is harmonic at every interior point \( P \) of \( R \) other than the \( P_i \) follows readily. About each of the \( P_i \) as center draw a small sphere \( S_i \) which does not contain \( P \). Then the region bounded by the \( S_i \) and the boundary of \( R \) is a region of the type considered in Theorem I from which it follows that \( f \) is harmonic at \( P \). It thus remains only to consider the nature of \( f \) in the neighborhood of any one of the \( P_i \).

In a paper presented to the Society, October 31, 1925, the writer has shown that if a function is harmonic at every point
in the deleted neighborhood of a point \( P \) it may be expressed in the form
\[
\frac{1}{r} + \Phi(x, y, z) + V(x, y, z)
\]
in this neighborhood. In this expression \( c \) is a constant, \( r \) the distance from \( P \) to \((x, y, z)\), \( V \) a function harmonic everywhere in the neighborhood of \( P \) as well as at \( P \) itself and \( \Phi \) a function harmonic in the deleted neighborhood and such that it is either identically zero or else there exist modes of approach to \( P \) for which \( \Phi \) will tend toward plus infinity and also modes of approach for which it will tend toward minus infinity; \( \Phi \) also possesses the property that its integral over the surface of any sphere with \( P \) as center vanishes.

Consider now two spheres \( S_1 \) and \( S_2 \) with center \( P \) and radii \( r_1 \) and \( r_2 < r_1 \). Apply Green's formula to the functions \( \Phi \) and \( 1/r - 1/r_1 \) for the region bounded by \( S_1 \) and \( S_2 \) and we have
\[
\int_{S_1} \left\{ \left( \frac{1}{r} - \frac{1}{r_1} \right) \frac{\partial \Phi}{\partial n} - \frac{\partial}{\partial n} \left( \frac{1}{r} - \frac{1}{r_1} \right) \Phi \right\} ds = 0,
\]
where the normal derivatives are taken toward the interior of the region \( S_1 \) \( S_2 \). Remembering that the integral of \( \Phi \) over any sphere with center \( P \) is zero and since \( 1/r - 1/r_1 \) is zero on \( S_1 \) and constant on \( S_2 \) we have from the above equation
\[
(3) \quad \int_{S_2} \frac{\partial \Phi}{\partial n} ds = 0.
\]
Since \( S_2 \) is any sphere interior to \( S_1 \) it follows that the integral of the normal derivative of \( \Phi \) over any sphere with center \( P \) and radius less than \( r_1 \) vanishes. The same result could of course be obtained from the property \( \int_{S}\Phi ds = 0 \) by considering the continuity of \( \Phi \) and using the theorem concerning the differentiation of a definite integral. In (3) because of the continuity of the first partial derivatives of \( \Phi \) in the deleted neighborhood of \( P \) we may take the normal derivative \textit{either} toward the interior or toward the
exterior normal of $S_2$ throughout the region of integration. Hence if the function

$$f = \frac{1}{r} + \Phi + V$$

is to be such that the integral of its normal derivative vanishes when taken over spheres in the neighborhood of $P$, the constant $c$ must be zero. Conversely we have shown by the above argument that if $c$ is zero $\int (\partial f / \partial n) ds$ will vanish when taken over every sphere with center $P$ and of radius $r < r_1$. We may now state the following theorem.

**Theorem II.** Every function which satisfies the conditions of § 2 in a region $R$ is harmonic at every interior point of $R$ except possibly at the points $P_i$. In the neighborhood of each $P_i$, $f$ is of the form $\Phi + V$. If $\Phi \equiv 0$, in the neighborhood of any $P_i$, $P_i$ is at most a removable discontinuity. If $\Phi \neq 0$, $f$ will be harmonic in the deleted neighborhood of $P$ and will be such that for certain modes of approach to $P$ it will tend toward plus infinity and for other modes to minus infinity.

It may be remarked in closing that although we have supposed $\int (\partial f / \partial n) ds$ to vanish only when taken over sufficiently small spheres with $P_i$ as center it is now easy to prove that it will vanish when taken over any regular surface $S$ in $R$ which does not pass through one of the $P_i$. We need merely to surround each $P_i$ in $S$ by a sphere $S_i$ lying entirely in $S$ and use the fact that

$$\int \frac{\partial f}{\partial n} ds = 0$$

where the integral is taken over $S$ and the spheres $S_i$. Then $\int_S (\partial f / \partial n) ds$ will vanish since the portion of (4) due to the $S_i$ vanishes.