TWO-WAY CONTINUOUS CURVES*

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A continuous curve $M$ will be said to be a two-way continuous curve, or to be "two-way continuous," provided it is true that between every two points of $M$ there exist in $M$ at least two arcs neither of which is a subset of the other. A point $P$ of a continuum $M$ is a cut point of $M$ provided it is true that the point set $M - P$ is not connected. Every point of a continuum $M$ which is not a cut point of $M$ will be called a non-cut point of $M$.

In a paper Concerning continua in the plane, among other results, I have established the following theorems which will be used in the proofs given in this paper.

I. If $K$ denotes the set of all the cut points of a continuum $M$, then every bounded, closed, and connected subset of $K$ is a continuous curve which contains no simple closed curve.

II. Every cut point of the boundary of a complementary domain of a bounded continuum $M$ is a cut point also of $M$.

III. If $K$, $H$, and $N$, respectively, denote the set of all the cut points, end points, and simple closed curves of a continuous curve $M$, then $K + H + N = M$.

IV. If $N$ denotes the point set consisting of all the simple closed curves contained in a continuous curve $M$, then every connected subset of $M - N$ is arcwise connected.

These results will be referred to by number as here listed. We shall now prove the following additional theorems.

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† Recently submitted for publication in the Transactions of this Society.
‡ For a definition of this term see R. L. Wilder, Concerning continuous curves, Fundamenta Mathematicae, vol. 7 (1925), p. 358.
THEOREM 1. In order that a continuous curve $M$ should be two-way continuous it is necessary and sufficient that every simple continuous arc of $M$ should contain a subarc which belongs to some single simple closed curve of $M$.

THEOREM 2. In order that a continuous curve $M$ should be two-way continuous it is necessary and sufficient that every arc of $M$ should contain a non-cut point of $M$.

Proof. The condition is sufficient. Let $A$ and $B$ denote any two points of a continuous curve $M$ which satisfies the condition. The curve $M$ contains one arc $t$ from $A$ to $B$. And from our hypothesis it follows that $t$ contains an interior point $O$ which is a non-cut point of $M$. It follows from a theorem of R. L. Moore's* that $M - O$ contains an arc $s$ from $A$ to $B$. Since $s$ does not contain the point $O$ of $t$, it follows that $t \neq s$, and therefore, that $M$ is two-way continuous.

The condition is also necessary. Let $t$ denote any definite arc of a two-way continuous curve $M$. By Theorem 1, $t$ contains a subarc $s$ which belongs to some simple closed curve $J$ of $M$. It is a consequence of a theorem of R. L. Moore's† that $J$ contains not more than a countable number of cut points of $M$. Since $s$ belongs to $J$ and contains uncountably many points altogether, it follows that $s$, and hence also $t$, must contain at least one non-cut point of $M$.

THEOREM 3. In order that a continuous curve $M$ should be two-way continuous it is necessary and sufficient that the set $K$ of all the cut points of $M$ should contain no continuum.

Proof. That the condition is sufficient is almost a direct consequence of Theorem 2. For, since by hypothesis $K$ can contain no continuum, therefore it can contain no arc.


† Concerning the cut points of continuous curves and of other closed and connected point sets, Proceedings of the National Academy, vol. 9 (1923), pp. 101–106, Theorem B*. 
Hence, every arc of \( M \) must contain a non-cut point of \( M \), and by Theorem 2, \( M \) is two-way continuous. The condition is also necessary. For suppose the set \( K \) of all the cut points of a two-way continuous curve \( M \) contains a continuum \( H \). Then by (I), \( H \) is a continuous curve. Hence, \( H \) contains at least one arc \( t \). But by Theorem 2, \( t \) must contain at least one non-cut point of \( M \). Thus the supposition that \( K \) contains a continuum leads to a contradiction.

**Theorem 4.** The boundary of every complementary domain of a two-way continuous curve is itself two-way continuous.

**Proof.** Let \( M \) denote the boundary of a complementary domain of a two-way continuous curve \( K \). Then \( M \) is a continuous curve.* Suppose, contrary to this theorem, that \( M \) is not two-way continuous. Then from Theorem 3 it follows that \( M \) must contain a continuum \( H \) every point of which is a cut point of \( M \). But by (II), every cut point of \( M \) is a cut point also of \( K \). And since \( K \) is two-way continuous, by Theorem 3, not every point of \( H \) can be a cut point of \( K \). Thus the supposition that \( M \) is not two-way continuous leads to a contradiction.

**Theorem 5.** If \( N \) denotes the point set consisting of all the simple closed curves contained in a two-way continuous curve \( M \), then \( M - N \) is totally disconnected.

**Proof.** Suppose \( M - N \) contains a connected set \( L \) consisting of more than one point. Then from (III) and (IV) it readily follows that \( L \) contains an arc \( t \) every point of which is a cut point of \( M \). But this is contrary to Theorem 2. It follows that \( M - N \) is totally disconnected.

**Theorem 6.** The boundary \( M \) of a complementary domain of a two-way continuous curve is the sum of two mutually exclusive point sets \( N \) and \( H \), where \( N \) is the sum of a countable

number of simple closed curves no two of which have more than one point in common, and $H$ is a totally disconnected set of points every one of which is a limit point of $N$ and is either a cut point or an end point of $M$.

Proof. By Theorem 4, $M$ is a two-way continuous curve. Let $G$ denote the collection of all the simple closed curves contained in $M$. R. L. Wilder* has shown that $G$ is countable and that no two curves of $G$ have more than one point in common. Let $N$ denote the point set obtained by adding together all the curves of the collection $G$. Then let $H$ denote the point set $M - N$. Since $M$ is two-way continuous, it readily follows that every point of $H$ is a limit point of $N$. By Theorem 5, $H$ is totally disconnected, and by (III), every point of $H$ is either a cut point or an end point of $M$. Hence, the sets $N$ and $H$ satisfy all the conditions of Theorem 6.

Theorem 7. In order that the boundary $M$ of a complementary domain $D$ of a continuous curve should be two-way continuous it is necessary and sufficient that $M$ should contain a point set $K$ such that (1) $D + K$ is uniformly connected im kleinen, and (2) every arc, if there be any, which $K'$ ($K$ plus all the limit points of $K$) contains, contains a non-cut point of $M$.

Proof. The condition is necessary. For let $K = M$. Clearly $D + K$ is uniformly connected im kleinen. And since $K'$ is two-way continuous, it follows by Theorem 2 that every arc of $K'$ contains a non-cut point of $M$. The condition is also sufficient. Let $M$ denote the boundary of a complementary domain $D$ of a continuous curve, and suppose that $M$ contains a point set $K$ satisfying conditions (1) and (2) in the statement of Theorem 7. Let $A$ and $B$ denote any two points of $M$. Now $M$ contains one arc $t$ from $A$ to $B$. Either $t$ is a subset of $K'$ or it is not. If $t$ is a subset of $K'$, then by hypothesis $t$ contains an interior

point $O$ which is not a cut point of $M$. Then by a theorem of R. L. Moore's, $M - O$ contains an arc from $A$ to $B$ which does not contain $O$, and which, therefore, is not a subset of $t$. Now if $t$ is not a subset of $K'$, then since $K'$ is closed, it readily follows that $t$ contains an arc $s$ which contains no point of $K'$. Let $X$ and $Y$ denote the end points and $O$ an interior point of $s$. Let $C$ be a circle having $O$ as center and not enclosing any point of $K$. Within $C$ and on $s$ there exist points $E$, $U$, $W$, and $G$ in the order $X$, $E$, $U$, $O$, $W$, $G$, $Y$. And within $C$ there exist arcs $EFG$ and $UVW$ having only their end points in common with $s$ and such that if $D_1$ and $D_2$ denote the interiors of the closed curves $UVWOU$ and $EUOWGFE$ respectively, then $D_1$ and $D_2$ are mutually exclusive domains each of which lies within $C$. Now since $D + K$ is uniformly connected in kleinien, and $C$ encloses no point of $K$, it readily follows that not both $D_1$ and $D_2$ can contain a subset of $D$ which has $O$ for a limit point. Hence, either $D_1$ or $D_2$ must contain a segment $QST$ of an arc $QST$ which has its end points on $s$ in the order $X$, $Q$, $O$, $T$, $Y$ and such that if $R$ denotes the interior of the closed curve $QOTSQ$, then $R$ contains no point whatever of $D + M$. Hence, $R$ lies wholly in some complementary domain $G$ of $D + M$. It follows from a theorem of R. L. Moore's† that the boundary $J$ of $G$ is a simple closed curve. The curve $J$ contains the arc $QOT$ of $t$. It follows that $M$ contains an arc from $A$ to $B$ which does not contain the point $O$ of $t$, and which, therefore, is not a subset of $t$. Hence, in any case, $M$ contains two arcs from $A$ to $B$ neither of which is a subset of the other, and therefore $M$ is two-way continuous.

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* Concerning continuous curves in the plane, loc. cit.
† Concerning continuous curves in the plane, loc. cit., Theorem 4.