ON THE FUNCTIONAL EQUATION

\[ f(x+y) = f(x) + f(y) \]

BY MARK KORMES

Fréchet,† and later Blumberg‡ and Sierpinski,§ have demonstrated that the solution of the functional equation

(1) \[ f(x + y) = f(x) + f(y) \]

which is measurable, has the form \( A \cdot x \), where \( A \) denotes a constant. In this note the following theorem is proved.

**THEOREM I.** Every solution of the functional equation (1) which is bounded on a set of positive measure is of the form \( A \cdot x \).

The proof depends on a theorem of Steinhaus¶ which can be stated as follows.

**LEMMA.** The set arising by arithmetic summation (addition of abscissas) of a set of positive measure, contains an interval.

Since \( f(x) \) is bounded on a set of positive measure, \( f(x+y) \) is bounded on an interval, and therefore \( f(x) \) must be of the form \( A \cdot x \) according to a theorem of Darboux.

**THEOREM Ia.** The statement of Theorem I remains true if \( f(x) \) is bounded on a set whose interior measure is positive.

If the interior measure of a set \( A \) is \( a > 0 \), then there exists** a measurable sub-set of \( A \) whose measure is equal to \( a \) (\( > 0 \)).

* Presented to the Society October 31, 1925.
‡ Blumberg, Convex functions, TRANSACTIONS OF THIS SOCIETY, vol. 20, p. 41.
|| The proof of this lemma will be a part of a paper entitled On arithmetic summation of point sets.
To this subset Theorem I can be applied and thus Theorem Ia is established.

Theorem I establishes a far more general condition than the one given by Fréchet, Blumberg, and Sierpinski. The following remarks show that this condition is incisive. The condition \( m^*(M) > 0 \) is essential, since there exist non-measurable solutions of (1) which are continuous on a set \( H \), where, for every interval \( \delta \),

\[
m_i(H \cdot \delta) = 0, \quad m_*(H \cdot \delta) = \delta.
\]

Let \( B \) denote a hamelian basis-set of all real numbers. If \( b \) is a number of \( B \), we define a solution of the functional equation (1) as follows:

\[
\begin{align*}
    f(x) &= 0, \quad \text{for the numbers of the set } (B - b); \\
    f(x) &= 1, \quad \text{for } x = b; \\
    f(x + y) &= f(x) + f(y), \quad \text{for all real numbers.}
\end{align*}
\]

In this way \( f(x) \) is completely defined. Let us denote by \( H \) the set of all points where \( f(x) = 0 \). If we denote by \( H^c \) the set of all numbers \( x + c \), where \( x \) assumes all values of \( H \) we have then \( H = H^{a \circ b} \), where the symbol \( \equiv \) means congruent, and \( a \) is a rational number. Then we have

\[
(H^{ab} \cdot H^{a' b}) = 0,
\]

if \( \alpha \neq \alpha' \), and

\[
K = \sum_{a} H^{a b},
\]

where \( K \) denotes the continuum. Therefore we must have, \( \dagger \) for every interval \( \delta \),

\[
m_i(H \cdot \delta) = 0, \quad m_*(H \cdot \delta) = \delta.
\]

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* The symbol \( m_i(M) \) shall signify the interior measure of \( M \), \( m_*(M) \) the exterior measure of \( M \).

\( \dagger \) For suppose \( m_i(H \cdot \delta) > 0 \). There must exist then a measurable subset \( P \subset H \) so that \( m(P) > 0 \). We would have \( m(P^{ab}) = m(P) > 0 \). On the other hand it can be shown easily, that then there exists a rational number \( \alpha_1 \), so that \( m(P^{ab} \cdot P) = \alpha > 0 \), but this is impossible, since \( (P^{ab} \cdot P) = 0 \) because \( P \subset H, P^{ab} \subset H^{a \circ b} \), and \( (H \cdot H^{a \circ b}) = 0 \). We must have therefore \( m_i(H \cdot \delta) = 0 \). See also M. Kormes, Treatise on basis-sets (Columbia University dissertation, not yet published), Theorem VIII.
Since the function $f(x)$ is everywhere 0 on the set $H$, it is bounded and continuous.

There exist non-measurable solutions of (1), which are continuous on a perfect set $P$, where $m(P)=0$. Let $P$ be the set of all numbers $z$ of the form

$$z = \frac{x_1}{10x_1!} + \frac{x_2}{(10^2x_2 + 10x_1)!} + \cdots$$

$$+ \frac{x_n}{(10^n x_n + 10^{n-1}x_{n-1} + \cdots + 10x_1)!} + \cdots,$$

where every $x_n$ is either 1 or 2. There cannot exist then any relation of the form

$$\sum_{\lambda} r_{\lambda} z_{\lambda} = 0$$

between the numbers $z$ of the set $P$, where $r_{\lambda}$ denotes a rational number, and in every case only a finite number of $r_{\lambda}$ are different from 0. The numbers of $P$ constitute a subset of a basis-set $B$ of all real numbers.† The existence of such basis-set was demonstrated in another paper.‡

We define now a solution of the functional equation (1) in the following way:

$$f(x) = 0, \text{ for all numbers of } P;$$

$$f(x) = 1, \text{ for all numbers of } B - P;$$

$$f(x+y) = f(x) + f(y) \text{ for all numbers of the continuum } K.$$ But this defines $f(x)$ completely, and it is clear that $f(x)$ is non-measurable and continuous on the perfect set $B$.

* M. Kormes, Treatise on basis-sets.

† To construct a basis-set $B$ which has a given set $P$ as a subset we proceed in the following way. We well-order the continuum $K$ in such a way that the numbers of $P$ precede all other numbers. The set $(K-P)$ is not empty, and since the set $P$ is not the entire basis-set of $K$, there must be a first number $a_1$ of $(K-P)$ which cannot be represented by numbers of $P$ in a linear way. If we consider the set $P_1 = P + a_1$ and reason in the same way as above, we obtain a basis-set $B$ of the continuum $K$. See also M. Kormes, Treatise on basis-sets.

‡ M. Kormes, loc. cit.
From Theorem I, the Fréchet-Sierpinski theorem* can be deduced immediately.

**Theorem II.** Every solution of (1) which is measurable has the form $A \cdot x$.

In fact, suppose that $f(x)$ is a solution of (1), and that $f(x)$ is measurable. Then there exists a perfect set $P$, where $m(P) > 0$, and $f(x)$ is continuous on $P$. Being finite, $f(x)$ must be bounded on $P$, and Theorem II is a simple consequence of Theorem I.

Theorem I can be generalized for functional equations in $n$ variables. A proof for two variables will be given below and it is quite analogous for $n(>2)$ variables.

**Theorem III.** Every solution of the functional equation

$$f(x + u, y + v) = f(x, y) + f(u, v)$$

where $x, y, u, v$ denote real numbers, which has the property that $f(x, 0)$ is bounded on a measurable set $M_x$, where $m(M_x) > 0$, and that $f(0, y)$ is bounded on a measurable set $M_y$, where $m(M_y) > 0$, has the form $A \cdot x + B \cdot y$.$\dagger$

We have

$$f(x, y) = f(x + 0, 0 + y) = f(x, 0) + f(0, y),$$

where $f(x, 0)$ is the solution of the functional equation

$$f(x + u, 0) = f(x, 0) + f(u, 0),$$

and $f(0, y)$ is the solution of the functional equation

$$f(0, y + v) = f(0, y) + f(0, v).$$

According to Theorem I, $f(x, 0)$ has the form $A \cdot x$, where $A = f(1, 0)$; and $f(0, y)$ has the form $B \cdot y$, where $B = f(0, 1)$. Therefore $f(x, y)$ has the form $A \cdot x + B \cdot y$.

* See second, third, and fourth footnotes on p. 689.

$\dagger$ We can assume also that $m_x (M_x) > 0$, and reason in a way similar to that indicated in the proof of Theorem Ia.
The same reasoning holds if \( f(x, a) \) and \( f(b, y) \) are bounded in \( M_x \) and \( M_y \), respectively. We have then \( f(x, a) = f(x, 0) + f(0, a) \) and \( f(b, y) = f(0, y) + f(b, 0) \). Hence \( f(x, 0) \) and \( f(0, y) \) must be therefore bounded in \( M_x \) and \( M_y \), respectively.

From Theorem III, the following theorem can easily be obtained.

**Theorem IV.** Every solution of the functional equation (2) which is bounded on a measurable set \( M_{xy} \), whose square measure is \( m(M_{xy}) > 0 \), has the form \( A \cdot x + B \cdot y \).

In order to prove this theorem let us suppose that

\[
m^{(2)}(M_{xy}) = a > 0.
\]

According to a theorem of Fubini, there must exist then a straight line \( y = b \) parallel to the \( X \)-axis, and a straight line \( x = a \) parallel to the \( Y \)-axis, so that \( m(M_{xa}) > 0 \) and \( m(M_{by}) > 0 \). Then \( f(x, a) \) would be bounded on the set \( M_{xa} \), where \( m(M_{xa}) > 0 \); and \( f(b, y) \) would be bounded on \( M_{by} \), where \( m(M_{by}) > 0 \). Therefore \( f(x, y) \) must have the form

\[
A \cdot x + B \cdot y.
\]

From Theorem IV, we may state the following theorem.

**Theorem V.** Every solution of the functional equation (2) which is measurable has the form \( A \cdot x + B \cdot y \).

If \( f(x, y) \) is a solution of (2) and it is measurable, then there exists a perfect set \( P \), where \( m^{(0)}(P) > 0 \), and \( f(x, y) \) is measurable on \( P \). Since \( f(x, y) \) is finite and \( P \) is closed, \( f(x, y) \) is bounded on \( P \), and we can apply Theorem IV.

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* Theorem of Fubini-Lebesgue; see de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, vol. 2 (2d ed.), pp. 117–120.
† Theorem of Steinhaus-Sierpinski; see Sierpinski, loc. cit.