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A CERTAIN CUBIC CURVE

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\( t = t_1s_1 \). Since \((t_1s_1)^n = s_1^n \) is in \((s)\), and \(n_1\) is prime to the order of \(s_1\), \(s_1\) is a power of \(s\). Thus \(t_1\), as well as \(t\), corresponds to \(t'\) in the isomorphism of \(G\) with \(G/(s)\); but \(t_1\) and \(t'\) are of the same order \(n_1\). Every element of \(G/(s)\) whose order is a divisor of \(m/m_1\) corresponds to an element of \(G\) whose order is a divisor of \(m\). It follows that \(t'\), and hence every element of \(G/(s)\) whose order divides \(n\), is commutative with every element whose order divides \(m/m_1\). Hence \(G/(s)\), being of order \(<mn\), contains an invariant subgroup of order \(n\). The corresponding subgroup of \(G\), being of order \(m_1n < mn\), also contains an invariant subgroup of order \(n\).

Thus in all cases \(G\) contains a subgroup of order \(n\). Similarly \(G\) contains a subgroup of order \(m\). \(G\) is evidently the direct product of these two subgroups.

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A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

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1. Introduction. If \(ABC, XYZ\) are two triangles, a cubic curve \(\Gamma_3\) may be associated with them as follows.* Let \((PQ, RS)\) denote the point of intersection of the lines \(PQ, RS\); then \(\Gamma_3\) is the locus of a point \(O\) such that \((OA, YZ), (OB, ZX), (OC, XY)\) are collinear and also the locus of a point \(O\) for which \((OX, BC), (OY, CA), (OZ, AB)\) are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take \(ABC\) as triangle of reference and let the points \(X, Y, Z\) have coordinates \((x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)\) respectively, then if \((\alpha, \beta, \gamma)\) are current coordinates

the condition for the collinearity of the second sets of three points may be expressed in the form

\[(\gamma x_1 - \alpha x_3)(\alpha y_2 - \beta y_1)(\beta z_3 - \gamma z_2) = (\alpha x_2 - \beta x_1)(\beta y_3 - \gamma y_2)(\gamma z_1 - \alpha z_3).\]

This equation represents a cubic curve passing through the points* \(A, B, C, X, Y, Z, (BC, YZ), (CA, ZX), (AB, XY), (CY, BZ), (AZ, CX), (BX, AY)\). There is evidently only one cubic curve passing through all these points, and since a cubic through all these points is obtained for the locus when the three points \((OA, YZ), (OB, ZX), (OC, XY)\) are collinear, \(\Gamma\) must be the locus in both cases.

This is easily verified analytically. If \(X_1, X_2, \ldots\) denote the co-factors of the constituents \(x_1, x_3, \ldots\) in the determinant

\[\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix},\]

the equation of the locus in the first case may be expressed in the form

\[\begin{vmatrix} \beta x_2 + \gamma x_3 & -\beta x_1 & -\gamma x_1 \\ -\alpha y_2 & \alpha y_1 + \gamma y_3 & -\gamma y_2 \\ -\alpha z_3 & -\beta z_3 & \alpha z_1 + \beta z_2 \end{vmatrix} = 0,\]

and the equivalence of the two equations is easily verified with the aid of a number of identities, one of which is

\[2X_1Y_2Z_3 - Y_1X_2Z_3 - Z_1X_2Y_3 = \Delta(2x_1y_2z_3 - y_1x_2z_2 - z_1x_2y_3) = \Delta(x_1X_1 + y_2Y_2 + z_3Z_3 - \Delta).\]

2. **Salmon’s Projective Invariant.** By a well known theorem due to Salmon the cross ratio of the four tangents to a cubic from the point on the curve is constant and represents

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* The properties of this configuration of 12 points and some degenerate cases of the cubic curve are discussed by H. M. Taylor, Proceedings of the London Society, (1), vol. 28 (1897), pp. 545–555.
an invariant of the curve. In the present case this invariant will be an invariant of the two triangles* and it seems worth while to calculate it. A direct calculation is laborious when use is made of the equation of the curve in one of the forms already given; the work may be simplified, however, by noticing that the four lines $BC, YZ, BZ, CY$ form a complete quadrilateral whose six vertices lie on the curve. Denoting the equations of these lines by $\alpha=0, \theta=0, \phi=0, \psi=0$ respectively, the equation of the curve may be written in the form

$$\frac{A}{\alpha} + \frac{B}{\theta} + \frac{C}{\phi} + \frac{D}{\psi} = 0,$$

where the identical relation between $\alpha, \theta, \phi, \psi$ is

$$\alpha + \theta + \phi + \psi = 0,$$

and

$$A = y_2z_2^2X_2^2Z_3, \quad B = y_1z_2X_1^2Y_2, \quad C = y_1^2z_2X_1^2, \quad D = x_1X_2^2 X_3^2.$$  

If $\lambda$ is one of the cross ratios of the four tangents, $\lambda$ is connected with the coefficients $A, B, C, D$ by the relation†

$$(\lambda + 1)^2(\lambda - 2)^2(\lambda - \frac{3}{2})^2J^3 = 27(\lambda^2 - \lambda + 1)^3I^3,$$

where $I$ and $J$, the usual invariants of a biquadratic equation, are given by the equations

$$12I = \Theta^2 - 48ABCD, \quad 216J = \Theta[72ABCD - \Theta^2],$$

$$I^3 - 27J^2 = A^2B^2C^2D^2[\Theta^4 - 64ABCD],$$

where

$$\Theta^2 = A^2 + B^2 + C^2 + D^2 - 2BC - 2CA - 2AB - 2AD - 2BD - 2CD.$$  

* A complete set of invariants of two triangles is given in a paper by D. D. Leib, Transactions of this Society, vol. 10 (1909), p. 361. In our case there is a correspondence between the vertices of the triangles and the invariant theory is a little different.

† The coefficient $m$ in the canonical equation $x^3+y^3+z^3+6mxyz=0$ is connected with $I$ and $J$ by the relation

$$\frac{64m^6(m^3-1)^3}{I^3} = \frac{(8m^3+20m^3-1)^2}{27J^2}.$$  

Substituting the expressions for $A$, $B$, $C$, $D$, we find that

$$
\Theta = y_1^2 z_1^2 x_2^4 x_3^4 \left[(x_3 y_1 z_2 - x_2 y_2 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3\right],

\quad A B C D = y_1^4 z_1^4 x_2^4 x_3^4 \left[x_1 X_1 y_2 Y_2 z_3 Z_3\Delta\right].
$$

The nature of the cubic is thus seen to depend on the value of the single invariant

$$
K = \frac{\left[(x_3 y_1 z_2 - x_2 y_2 z_1)^2 - 4\Delta x_1 y_2 z_3 - 4X_1 Y_2 Z_3\right]^2}{x_1 y_2 z_3 X_1 Y_2 Z_3\Delta}.
$$

The cubic has a double point when this invariant is either infinite or 64. When $\Theta = 0$ we have $J = 0$ and the four tangents form a harmonic pencil. If $A$, $B$, $C$, $Y$, and $Z$ are given, the locus of $X$ when $\Theta = 0$ is a conic. This, however, is not the complete locus when the tangents form a harmonic pencil because $J$ vanishes also when $\Theta^2 = 72 A B C D$ or $K = 72$. It should be noticed that the expression $K$ remains the same when we replace each constituent in the determinant by its co-factor. We have in fact an identity of type

$$
X_3 Y_1 Z_2 - X_2 Y_3 Z_1 = \Delta(x_3 Z_2 - y_2 Y_3 + x_2 X_2 - x_2 Y_2)
$$

$$
= \Delta(x_3 y_1 z_2 - x_2 y_2 z_1),
$$

with the aid of which the preceding remark is easily verified.

Quantities that possess this property may be called invariants of the determinant. To find such quantities we may commence by finding pseudo-invariants, that is, quantities which retain the same form except for a power of the determinant. The quantities $x_1 X_1$, $x_2 X_2$, $x_3 X_3$, $y_1 Y_1$, $y_2 Y_2$, $y_3 Y_3$, $z_1 Z_1$, $z_2 Z_2$, $z_3 Z_3$ all possess this last property when considered as functions of the constituents $x_1$, $x_2$, etc. and a similar remark may be made in the case of a determinant of higher order. For the general case of a determinant of order $n$ I do not know definitely how many pseudo-invariants of degree $n$ in the constituents are linearly independent. A consideration of the cases $n = 2, 3, 4$ leads to the conclusion that this number $N$ may be given by the formula

$$
N = (n - 1)^2 + 1.
$$
When \( n = 2 \), it is evident that \( N = 2 \). When \( n = 3 \) it appears that \( N = 5 \), for each of the nine quantities \( x_1X_1, x_2X_2, x_3X_3, y_1Y_1, y_2Y_2, y_3Y_3, z_1z_1, z_2z_2, z_3z_3 \) can be expressed as a linear function of the five linearly independent quantities \( x_1X_1, x_2X_2, y_1Y_1, y_2Y_2, \Delta \).

In the case of a determinant of the fourth order

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 & x_4 \\
  y_1 & y_2 & y_3 & y_4 \\
  z_1 & z_2 & z_3 & z_4 \\
  t_1 & t_2 & t_3 & t_4 \\
\end{vmatrix},
\]

the problem is more complex because in addition to the pseudo-invariants \( x_1X_1, \ldots \) etc. there are pseudo-invariants of type \( (y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1) \).

Adopting the notation

\[
\begin{pmatrix}
  y_2 \\
  x_4 \\
\end{pmatrix}_{14} = (y_2z_3 - y_3z_2)(x_1t_4 - x_4t_1) = \begin{pmatrix}
  x_2 \\
  x_4 \\
\end{pmatrix}_{23},
\]

we observe that the relations of type

\[
\begin{align*}
y_1Y_1 &= x_2X_2 + \begin{pmatrix}
  x_2 \\
  x_4 \\
\end{pmatrix}_{12} + \begin{pmatrix}
  z_2 \\
  z_4 \\
\end{pmatrix}_{12} - \begin{pmatrix}
  x_4 \\
  y_4 \\
\end{pmatrix}_{31} - \begin{pmatrix}
  x_4 \\
  y_4 \\
\end{pmatrix}_{14}, \\
y_2Y_2 &= x_1X_1 + \begin{pmatrix}
  x_2 \\
  x_4 \\
\end{pmatrix}_{12} + \begin{pmatrix}
  z_2 \\
  z_4 \\
\end{pmatrix}_{12} + \begin{pmatrix}
  x_4 \\
  y_4 \\
\end{pmatrix}_{24} + \begin{pmatrix}
  x_4 \\
  y_4 \\
\end{pmatrix}_{23}, \\
y_3Y_3 &= x_4X_4 + \begin{pmatrix}
  x_2 \\
  x_4 \\
\end{pmatrix}_{34} + \begin{pmatrix}
  z_2 \\
  z_4 \\
\end{pmatrix}_{34} - \begin{pmatrix}
  x_2 \\
  y_2 \\
\end{pmatrix}_{31} - \begin{pmatrix}
  x_2 \\
  y_2 \\
\end{pmatrix}_{23}, \\
y_4Y_4 &= x_3X_3 + \begin{pmatrix}
  x_2 \\
  x_4 \\
\end{pmatrix}_{32} + \begin{pmatrix}
  z_2 \\
  z_4 \\
\end{pmatrix}_{32} - \begin{pmatrix}
  x_2 \\
  y_2 \\
\end{pmatrix}_{14} - \begin{pmatrix}
  x_2 \\
  y_2 \\
\end{pmatrix}_{24}
\end{align*}
\]

indicate that all the pseudo-invariants of type \( y_sY_s, z_sZ_s, t_sT_s \)

can be expressed in terms of the pseudo-invariants \( x_1X_1, x_2X_2, x_3X_3, x_4X_4 \) and the 18 quantities of type

\[
\begin{pmatrix}
  y_s \\
  x_4 \\
\end{pmatrix}_{14}.
\]
It would be wrong, however, to suppose that there are 22 linearly independent pseudo-invariants of degree four, for there are the six identities

\[
x_1X_1 + x_4X_4 = (xy)_{23} + (zx)_{23} + (xf)_{23},
\]

\[
x_2X_2 + x_4X_4 = (xy)_{31} + (zx)_{31} + (xf)_{31},
\]

\[
x_3X_3 + x_4X_4 = (xy)_{12} + (zx)_{12} + (xf)_{12},
\]

\[
x_2X_2 + x_3X_3 = (xy)_{14} + (zx)_{14} + (xf)_{14},
\]

\[
x_2X_2 + x_1X_1 = (xy)_{24} + (zx)_{24} + (xf)_{24},
\]

\[
x_1X_1 + x_2X_2 = (xy)_{34} + (zx)_{34} + (xf)_{34},
\]

three relations of type

\[
\Delta = (xy)_{34} + (xy)_{14} + (xy)_{24} + (xy)_{23} + (xy)_{31} + (xy)_{12},
\]

three relations of type

\[
\Delta = (yz)_{12} + (zx)_{12} + (xf)_{12} + (yf)_{12} + (zf)_{12},
\]

and the relation \( x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 = \Delta \). The number of linearly independent pseudo-invariants thus seems to be ten.