INTEGERS REPRESENTED BY POSITIVE TERNARY QUADRATIC FORMS

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1. Introduction. Dirichlet† proved by the method of §2 the following two theorems:

THEOREM I. \( A = x^2 + y^2 + z^2 \) represents exclusively all positive integers not of the form \( 4^k(8n+7) \).

THEOREM II. \( B = x^2 + y^2 + 3z^2 \) represents every positive integer not divisible by 3.

Without giving any details, he stated that like considerations applied to the representation of multiples of 3 by \( B \). But the latter problem is much more difficult and no treatment has since been published; it is solved below by two methods.

Ramanujan‡ readily found all sets of positive integers \( a, b, c, d \) such that every positive integer can be expressed in the form \( ax^2 + by^2 + cz^2 + du^2 \). He made use of the forms of numbers representable by

\[
\begin{align*}
A, B, C &= x^2 + y^2 + 2z^2, \\
D &= x^2 + 2y^2 + 2z^2, \\
E &= x^2 + 2y^2 + 3z^2, \\
P &= x^2 + 2y^2 + 4z^2, \\
G &= x^2 + 2y^2 + 5z^2.
\end{align*}
\]

He gave no proofs for these forms and doubtless obtained his results empirically. We shall give a complete theory for these forms. These cases indicate clearly methods of procedure for any similar form.

For a new theorem on forms in \( n \) variables, see §9.

* Presented to the Society, December 31, 1926.
‡ Proceedings of the Cambridge Philosophical Society, vol. 19 (1916-19), pp. 11-15. He overlooked the fact that \( x^2 + 2y^2 + 5z^2 + 5u^2 \) fails to represent 15.
2. The Form B. Let $B$ represent a multiple of 3. Since $-1$ is a quadratic non-residue of 3, $x$ and $y$ must be multiples of 3. Thus $B = 3\beta$, $\beta = 3x^2 + 3y^2 + z^2$. Since $\beta \equiv 0$ or $1 \pmod{3}$, $\beta$ represents no integer $3n+2$. If $\beta$ is divisible by 3, $z$ is divisible by 3 and $B$ is the product of a like form by 9. We shall prove that $\beta$ represents every positive integer $3n+1$. These results and Theorem II give

**Theorem III.** $x^2 + y^2 + 3z^2$ represents exclusively all positive integers not of the form $9k(9n+6)$.

We shall change the notation from $\beta$ to $f$ and employ the fact that the only reduced positive ternary forms of Hessian 9 are*

$$
\begin{align*}
    f &= x^2 + 3y^2 + 3z^2, \\
    g &= x^2 + y^2 + 9z^2, \\
    h &= x^2 + 2y^2 + 5z^2 - 2yz, \\
    l &= 2x^2 + 2y^2 + 3z^2 - 2xy.
\end{align*}
$$

No one of $g$, $h$, $l$ represents an integer $8m+7$. For $g$ this follows from Theorem I, since $g \equiv A \pmod{8}$. Suppose $l \equiv 7 \pmod{8}$. Then $z$ is odd, $2s \equiv 4 \pmod{8}$, where $s = x^2 + y^2 - xy$. Thus $s$ is even and $(1+x)(1+y) \equiv 1 \pmod{2}$, $x$ and $y$ are even, and $s \equiv 0 \pmod{4}$, a contradiction. Finally, let $h \equiv 7 \pmod{8}$. If $y$ is even, $h \equiv x^2 + z^2 \equiv 3 \pmod{4}$. Hence $y$ is odd and

$$
3 \equiv h \equiv x^2 + (z-1)^2 + 1 \pmod{4},
$$

so that $x$ and $z-1$ are odd. Write $z = 2Z$. Then $h \equiv 3 + 4Z (Z-1) \equiv 3 \pmod{8}$.

Consider the ternary form lacking the term $xy$:

$$
(1) \quad \phi = ax^2 + by^2 + cz^2 + 2ryz + 2szx.
$$

Its Hessian $H$ is $a\Delta - bs^2$, where $\Delta = bc - r^2$. Take $H = 9$, $s = 1$, $\Delta = 24t$, $t = 6k+1$. Then $b = 3\beta, \beta = 8at - 3$. If $a$ is not divisible by 3, $\beta = 48ak + 8a - 3$ is a linear function of $k$ with relatively prime coefficients and hence represents an infinitude of primes.

* Eisenstein, Journal für Mathematik, vol. 41 (1851), p. 169. By the Hessian $H$ of $\phi$ we mean the determinant whose elements are the halves of the second partial derivatives of $\phi$ with respect to $x, y, z$. Eisenstein called $-H$ the determinant of $\phi$. The facts borrowed in this paper from Eisenstein's table have been verified independently by the writer.
Take \(a = 3n + 1\). Then \(\beta \equiv -1 (\text{mod } 6)\),

\[
\begin{align*}
\left( \frac{-3}{\beta} \right) &= -1, & \left( \frac{2}{\beta} \right) &= \frac{1}{1}, \\
\left( \frac{t}{\beta} \right) &= \left( \frac{\beta}{t} \right) = \left( \frac{-3}{t} \right) = 1, & \left( \frac{-\Delta}{\beta} \right) &= 1.
\end{align*}
\]

Hence \(w^2 \equiv -\Delta (\text{mod } \beta)\) is solvable. We can choose a multiple \(r\) of \(3\) such that \(r \equiv w (\text{mod } \beta)\). Then \((\Delta + r^2)/b\) is an integer \(c\). Since \(\phi\) represents \(b \equiv 7 (\text{mod } 8)\), it is equivalent to no one of \(g, h, l\) and hence is equivalent to \(f\). Thus \(f\) represents every \(a = 3n + 1\).

**Theorem IV.** \(x^3 + 3y^2 + 3z^2\) represents exclusively all positive integers not of the form \(9k(3n + 2)\).

This theory for \(B\) made use of forms of the larger Hessian \(9\). We shall next show how to deduce a theory making use only of forms having the same Hessian \(3\) as \(B\).

3. A New Theory for \(B\). We proved that \(f\) is equivalent to a form (1) having \(a = 3n + 1, \ b = 3\beta, \ r = 3\rho, \ s = 1\), where \(\beta = 8at - 3\) is a prime. In \(9 = H = a(bc - r^2) - 3\beta\), replace \(\beta\) by its value. Thence \(c = (8t + 3\rho^2)/\beta \equiv 1 (\text{mod } 3), \ c = 1 + 3\gamma\). In (1) replace \(x\) by \(X - \varepsilon\). We get

\[
\psi = aX^2 - 6nXz + 3(n + \gamma)z^2 + 3\beta y^2 + 6\rho yz.
\]

Write \(3z = Z, \ 3y = Y\). Then

\[
3\psi = 3aX^2 - 6nXZ + (n + \gamma)Z^2 + \beta Y^2 + 2\rho YZ
\]

is equivalent to \(3f = 3x^2 + Y^2 + Z^2\) and hence to \(B\). In \(3\psi\), replace \(a\) by \(\alpha\), \(\beta\) by \(b, \rho\) by \(r\). We conclude that (1) is equivalent to \(B\) if

\[
(3) \quad a = 3\alpha, \quad \alpha = 3n + 1, \quad b = 8\alpha t - 3, \quad t = 6k + 1, \quad s = -3n = 1 - \alpha.
\]

We shall now give a direct proof that there exists a form (1) of Hessian 3 which satisfies conditions (3) and is equivalent to \(B\). In \(H = 3\), replace \(a\) and \(s\) by their values in (3). We get

\[
\begin{align*}
b + 3 + 3\alpha r^2 - abP &= 0, & P &= 3c + 2 - \alpha.
\end{align*}
\]
Replace the first term \( b \) by its value in (3), and cancel \( \alpha \). We get

\[
8t + 3r^2 - bP = 0, \quad -24t \equiv (3r)^2 \pmod{b}.
\]

This congruence is solvable by (2) with \( \beta \) replaced by \( b \). By (4),

\[
8t(1-P) \equiv 0, \quad P \equiv 1 \pmod{3}.
\]

Hence the value of \( \alpha \) determined by \( P \) is an integer. The only two reduced forms of Hessian 3 are \( B \) and \( \chi = x^2+2\sigma \), where \( \sigma = y^2+yz+z^2 \). Suppose \( \chi \equiv 5 \pmod{8} \). Then \( x \) is odd and \( \sigma \equiv 2 \pmod{4} \). Thus

\[
(1 + y)(1 + z) \equiv 1, \quad y \equiv z \equiv 0 \pmod{2}, \quad \sigma \equiv 0 \pmod{4}.
\]

This contradiction shows that \( b \) is not represented by \( \chi \). Since (1) represents \( b \), it is not equivalent to \( \chi \) and hence is equivalent to \( B \). Thus \( B \) as well as \( \phi \) represents\(^* \) \( a = 3\alpha \). This completes the new proof of Theorem III by using only forms of Hessian 3. The numbers represented by \( \chi \) are given by Theorem XI.

4. The Form \( C = x^2+y^2+2z^2 \). By Theorem I, \( A \) represents every positive \( 4k+2 \). Then just two of \( x, y, z \) are odd, say \( x \) and \( y \), while \( z = 2Z \). Then \( x = X + Y, y = X - Y \) determine integers \( X \) and \( Y \). Hence

\[
X^2 + Y^2 + 2Z^2 = 2k + 1,
\]

so that \( C \) represents all positive odd integers.\(^\dagger \) If

\[
m = 4^k(8n+7), m = x^2 + Y^2 + z^2,
\]

by Theorem I. Hence

\[
2m = (X + Y)^2 + (X - Y)^2 + 2z^2.
\]

Conversely, if \( C \) is even, it is of the latter form.

**Theorem V.** \( C \) represents exclusively all positive integers not of the form \( 4^k(16n+14) \).

5. The Form \( D = x^2 + 2y^2 + 2z^2 \). If \( m \) is odd and \( \neq 8n+7 \), \( m = x^2 + Y^2 + Z^2 \) by Theorem I. Then \( x + Y + Z \) is odd. Permuting, we may take \( x \) odd, and write \( Y + Z = 2y, Y - Z = 2z \).

\* If we apply the method of §2 when \( H = 3 \) and hence take \( s = 1, b = 3\beta \), where \( \beta \) is a prime \( = -1 \pmod{8} \), we find that it fails for all choices of \( \Delta \).

\^Lebesque, Journal de Mathématiques, (2), vol. 2 (1857), p. 149, gave a long proof by the method of §2.
Then \( m = D \). Next, let \( m = 2r \) be any even integer not of the form \( 4^k(8n+7) \). Then \( r \neq 4^i(16n+14) \). By Theorem V, \( r \) is represented by \( C \). Then \( m = 2r \) is represented by \( D \) with \( x \) even.

**Theorem VI.** \( D \) represents exclusively* all positive integers not of the form \( 4^k(8n+7) \).

6. *The Form \( F = x^2 + 2y^2 + 4z^2 \).* Every odd integer is represented by \( C \) with \( x+y \) odd, whence one of \( x \) and \( y \) is even. Any integer \( \neq 4^k(8n+7) \) is represented by \( D \), and \( 2D = (2y)^2 + 2x^2 + 4z^2 \).

**Theorem VII.** \( F \) represents exclusively† all positive integers not of the form \( 4^k(16n+14) \).

The simple methods used in proving Theorems V–VII apply also to \( x^2 + 2y + 2z \) when \( r \) and \( s \) are both \( \leq 3 \), and when \( r = 1 \) or \( 3, s = 4 \).

7. *The Form \( G = x^2 + 2y^2 + 5z^2 \).* The only reduced forms of Hessian 10 are \( G, J = x^2 + y^2 + 10z^2, K = 2x^2 + 2y^2 + 3z^2 + 2xz \), and \( L = 2x^2 + 2y^2 + 4z^2 + 2yz + 2xz + 2xy \). Neither \( J \) nor \( K \) represents a number of the form \( 2(8n+3) \). For, if \( K \) is even, \( s = 2Z, K = 2M, M = X^2 + y^2 + 5Z^2 \), where \( X = x+Z \). Since \( M \) is congruent to a sum of three squares modulo 4, it is congruent to 3 if and only if each square is odd, and then \( M \equiv 7 (\text{mod } 8) \). If \( J \) is even, \( x = y + 2t, J = 2N, N = (y + t)^2 + t^2 + 5z^2 \equiv 3 (\text{mod } 8) \).

We now apply the method of §2 to prove that \( G \) represents every positive integer prime to 10. Take \( \Delta = 16k, k = 10l \pm 3 \). Then \( b = 2\beta, \) where \( \beta = 8ka - 5 \) represents infinitely many primes. Now

\[
\left( \frac{-\beta}{k} \right) = \left( \frac{5}{k} \right) = \left( \frac{k}{5} \right) = -1,
\]

\[
\left( \frac{-\Delta}{\beta} \right) = - \left( \frac{k}{\beta} \right) = - \left( \frac{-\beta}{k} \right) = 1.
\]

Since (1) represents \( b \), which is of the form \( 2(8n+3) \), it is not equivalent to \( J \) or \( K \). Since it represents the odd \( a \), it is not

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* \( D = x^2 + (y+z)^2 + (y-z)^2 \neq 4^k(8n+7) \) by Theorem I.
† If \( F \) is even, \( x \) is even and \( F \) is the double of a form \( D \).
equivalent to $L$. Hence (1) is equivalent to $G$, which therefore represents $a$.

If $G$ represents a multiple of 5, it is the product of 5 by $g = 5x^2 + 10y^2 + z^2$, whence $G$ represents no $5(5n \pm 2)$. Also, $g$ is divisible by 5 only when $z$ is. Thus $G$ is divisible by 25 only when it is a product of 25 by a form like $G$.

To prove* that $G$ represents every $5\alpha$ if $\alpha = 5n \pm 1$ is odd, employ (1) with $a = 5\alpha$, $b = 2\beta$, $\beta = 8\alpha t - 5$, $r = 2p$, $s = 1 \pm \alpha$.

The Hessian of (1) is 10 if

$$\beta + 5 + 10\alpha \rho^2 - \alpha \beta P = 0, \quad P = 5c \pm 2 - \alpha.$$

Take $t$ prime to 10 and replace the first $\beta$ by its value. Thus $8t + 10\rho^2 - \beta P = 0$, $P \equiv \pm 1 \pmod{5}$. Hence $P$ yields an integral value for $c$. Also,

$$\left(\frac{t}{\beta}\right) = \left(\frac{-\beta}{t}\right) = \left(\frac{5}{t}\right),$$

$$\left(\frac{5}{\beta}\right) = \left(\frac{\beta}{5}\right) = \left(\frac{\pm 8t}{5}\right) = -\left(\frac{t}{5}\right),$$

$$\left(\frac{-80t}{\beta}\right) = -\left(\frac{5t}{\beta}\right) = 1.$$

Next, if $G$ is even, $x = z + 2w$ and $G = 2T$, where

$$T = y^2 + 2w^2 + 2wz + 3z^2, \quad S = x^2 + y^2 + 5z^2$$

are the only reduced forms of Hessian 5. Every positive integer $a$ prime to 5 is represented by $T$. Take $\Delta = 8k, k = 10m \pm 1$.

Then $b = a\Delta - 5$ represents an infinitude of primes, and

$$\left(\frac{-2}{b}\right) = 1, \quad \left(\frac{-\Delta}{b}\right) = \left(\frac{k}{b}\right) = \left(\frac{-b}{k}\right) = \left(\frac{5}{k}\right) = \left(\frac{k}{5}\right) = 1.$$

Now $b = 3 \pmod{16}$ is not represented by $S$, as proved for $M$. Hence (1) is equivalent to $T$ and not $S$. Thus $T$ represents $a$.

* Or we may use the method of §2. Of the ten properly primitive reduced forms of Hessian 50, all except $g$ fail to represent numbers $\equiv 14 \pmod{16}$. To prove that $g$ represents $\alpha = 5n \pm 1$ when odd, take $\Delta = 80t$, whence $b = 10\beta, \beta = 8\alpha t - 5$; apply (5). From this proof was reconstructed the shorter one in the text.
We saw that \(2T = G\) represents no \(5(5m \pm 2)\). Thus \(T\) represents no \(5(5n \pm 1)\). To prove that \(T\) represents every \(5\alpha\), where \(\alpha = 5n \pm 2\), employ (1) with \(a = 5\alpha, s = 1 \pm 2\alpha\). Its Hessian is 5 if
\[
b + 5 + 5\alpha r^2 - abP = 0, \quad P = 5c - 4\alpha \pm 4.
\]
Replace the first term \(b\) by \(8t\alpha - 5\), where \(t\) is prime to 10. Thus \(8t + 5r^2 - bP = 0\). Hence \(8t(1 \mp 2P) = 0, P = \pm 3(\text{mod } 5)\), and the resulting value of \(c\) is an integer. Also
\[
\left(\frac{5}{b}\right) = \left(\frac{b}{5}\right) = \left(\frac{\pm 16t}{5}\right) = \left(\frac{t}{5}\right),
\]
\[
\left(\frac{t}{b}\right) = \left(\frac{-b}{t}\right) = \left(\frac{5}{t}\right), \quad \left(\frac{-40t}{b}\right) = 1.
\]
We have now proved the two theorems:

**Theorem VIII.** \(T\) represents exclusively all \( \neq 25^k(25n \pm 5)\).

**Theorem IX.** \(G\) represents exclusively all \( \neq 25^k(25n \pm 10)\).

8. *The Form \(E = x^2 + 2y^2 + 3z^2\).* We shall outline a proof of the following theorem.

**Theorem X.** \(E\) represents exclusively all \( \neq 4^k(16n + 10)\).

The only reduced forms of Hessian 6 are \(E\) and
\[
Q = x^2 + y^2 + 6z^2, \quad R = 2x^2 + 2y^2 + 2z^2 + 2xy.
\]
To prove that \(E\) represents every positive odd integer \(a\), take \(\Delta = 9k, k = 8t + 3\). Then \(b = 3\beta\), where \(\beta\) represents primes. Also \((-\Delta/\beta) = 1\). The resulting form (1) represents the odd \(a\) and hence is not equivalent to \(R\). Since it represents \(b = 3(3n + 1)\), it is not equivalent to \(Q\). For, if \(Q\) is divisible by 3, both \(x\) and \(y\) are.

If \(E\) is even, then \(x = x + 2t\) and \(E = 2U,\)
\[
U = y^2 + 2z^2 + 2zt + 2t^2
\]
and conversely. In place of \(U\) we employ the like form \(\chi\) of §3. To show that \(\chi\) represents \(a = 2\alpha\) when \(\alpha\) is odd, take \(\Delta = 9k, k = 4t + 1\). Then \(b = a\Delta - 3 \equiv 6(\text{mod } 9)\) is not represented by the remaining reduced form \(B\) of Hessian 3 (Theorem III). Also \(b = 3\beta, (-\Delta/\beta) = +1\).
Finally, \( \chi \) represents every positive odd integer \( a \neq 5 \pmod{8} \). Write \( a = \frac{1}{2}(3a-1) \) and take \( \Delta = 9k, \ k = 2h+1 \). Then \( b = 6q, \ q = 3ah + \alpha \). If \( a = 8A + 1 \), take \( h = 4t, \ t \) odd. Then \( q = 12Ak + 12t+1 \),

\[
\left( \frac{-\Delta}{q} \right) = \left( \frac{k}{q} \right) = \left( \frac{q}{k} \right) = \left( \frac{12t + 1 - k}{k} \right) = \left( \frac{4t}{k} \right) = \left( \frac{k}{t} \right) = 1.
\]

If \( a = 8A + 3 \), take \( h = 4t+1 \). If \( a = 8A + 7 \), take \( h = 4t+1 \). In each case \( (-\Delta/q) = 1 \). In all three cases, \( q \) represents an infinity of primes.

**Theorem XI.** \( x^2 + 2y^2 + 2yz + 2z^2 \) represents exclusively all positive integers not of the form \( 4^k(8n+5) \).

9. *Forms in \( n \) Variables.* By a simple modification of Ramanujan’s determination of quaternary forms which represent all positive integers, we readily prove*

**Theorem XII.** If, for \( n \geq 5 \), \( f = a_1x_1^2 + \cdots + a_nx_n^2 \) represents all positive integers, while no sum of fewer than \( n \) terms of \( f \) represents all positive integers, then \( n = 5 \) and

\[
f = x^2 + 2y^2 + 5z^2 + 5u^2 + ev^2, \quad (e = 5, 11, 12, 13, 14, 15),
\]

and these six forms \( f \) actually have the property stated.

After this paper was in type, I saw that J. G. A. Arndt gave† the Dirichlet type of proof which appears in §2 above, but not the improved new proof of §3. For the form \( G \) of §7, he treats only numbers not divisible by 5.