A NEW CHARACTERIZATION OF PLANE CONTINUOUS CURVES*

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A number of authors† have given necessary and sufficient conditions that a bounded continuum be a continuous curve. However new conditions are always of interest as no one characterization applies without difficulty to all problems. It is the purpose of this paper to give a new necessary and sufficient condition that a bounded plane continuum be a continuous curve. Also this gives a condition under which a subcontinuum of a continuous curve is itself a continuous curve. Finally we prove a new property of continuous curves.

THEOREM I. In order that a continuum \(N\), which is a subset of a plane continuous curve \(M\) and such that \(M - N\) consists of a finite number of maximal connected subsets‡, be a continuous curve, it is necessary and sufficient that if \(P_1, P_2, P_3, \ldots\) is any sequence of distinct points of a maximal connected subset of \(M - N\) which has a sequential limit point \(P\), then there exists an increasing sequence of positive integers \(n_1, n_2, n_3, \ldots\)

* Presented to the Society, October 30, 1926.
‡ A point set \(K\) which is a subset of a point set \(M\) is said to be a proper subset of \(M\) if \(M - K\) is not vacuous. A connected subset \(K\) of a point set \(M\) is said to be a maximal connected subset of \(M\) if \(K\) is not a proper subset of any connected subset of \(M\).
and a set of arcs of $M - N$, $P_{n_1}P_{n_2}$, $P_{n_2}P_{n_3}$, \ldots, such that the set $P + \sum_{i=1}^{\infty} P_{n}P_{n_{i+1}}$ is closed.

**Proof.** A. The condition is necessary. Let $P_1, P_2, P_3, \ldots$ be any sequence of points of a maximal connected subset $D$ of $M - N$ which has a sequential limit point $P$. There are two cases to consider.

(a). If $P$ is a point of $M - N$, $D$ contains $P$ and there exists a circle $C_1$ with center at $P$ which encloses no point of $N$. We may suppose that for every $i$, $P_i \not\subseteq P$, for if any $P_i$ were $P$ we could drop this point from the sequence and consider the remainder. Since $M$ is connected im kleinen, there exists a circle $C_2$ with center at $P$ such that $r_i \leq r_i/2$, where $r_i$ denotes the radius of $C_i$, and such that every point of $M$ in the interior of $C_2$ can be joined to $P$ by an arc* of $M$ which lies wholly in the interior of $C_1$. Let $n_1$ be the smallest integer so that $P_{n_1}$ is interior to $C_2$. In general there exists a circle $C_{i+1}$ with center at $P$ such that $r_{i+1} \leq r_i/2$ and $P_{n_{i-1}}$ lies in the exterior of $C_{i+1}$ and such that every point of $M$ in the interior of $C_{i+1}$ can be joined to $P$ by an arc of $M$ which lies wholly in the interior of $C_i$. Let $n_i$ be the smallest integer such that $P_{n_i}$ lies in the interior of $C_{i+1}$ and let $P_{n_i}P$ denote the arc of $M$ (actually of $M - N$) whose existence is shown above. For every $i$, the set $P_{n_i}P + P_{n_{i+1}}P$ contains an arc $P_{n_i}P_{n_{i+1}}$ from $P_{n_i}$ to $P_{n_{i+1}}$. Since every arc $P_{n_i}P_{n_{i+1}}$ lies in the interior of the circle $C_i$ and the numbers $r_i$ approach 0 as $i$ increases, the set $P + \sum_{i=1}^{\infty} P_{n_i}P_{n_{i+1}}$ is closed.

(b). If $P$ is a point of $N$, let $C_1$ be a circle with center at $P$ and radius $r$ so small that $N$ and $D$ contain points exterior to $C_1$. This is possible unless $N$ is identical with $M$ and in this case our theorem is obvious. Let $D_{11}, D_{12}, D_{13}, \ldots$ be the maximal connected subsets of $D \cdot I(C_1)$.

* That this can be done by an arc, see J. R. Kline, *Concerning the approachability of simple closed and open curves*, Transactions of this Society, vol. 21 (1920), page 453 and footnote.

† If $C$ is a circle, $I(C)$ denotes the interior of $C$. If $A$ and $B$ are point sets, $A \cdot B$ denotes the set of points common to $A$ and $B$. 
We shall show that one of these sets, which we will denote by $D_1$, contains infinitely many of the points $P_i$. If this is not true, then if $C_2$ denotes the circle with center at $P$ radius $r/2$, for infinitely many values of $i$, $D_{1i}$ has a point within $C_2$ and $C_1$ contains a limit point of $D_{1i}$. Thus infinitely many of the sets $D_{1i}$ are of diameter greater than $r/4$. But this contradicts the theorem that if $M+C_1$ and $N+C_1$ are continuous curves and $N+C_1$ is a subset of $M+C_1$ then $M+C_1-(N+C_1)$ cannot contain more than a finite number of maximal connected subsets of diameter greater than $r/4$.*

Let $n_1$ be the smallest integer such that $D_1$ contains $P_{n_1}$. Similarly one, $D_2$, of the maximal connected subsets of $D_1 \cdot I(C_2)$ contains infinitely many of the points $P_i$. Let $n_2$ be the smallest integer greater than $n_1$ such that $D_2$ contains $P_{n_2}$. In general let $C_j(j=1, 2, 3, \cdots)$ be a circle with center at $P$ and radius $r/j$ and let $D_j$ be a maximal connected subset of $D_{j-1} \cdot I(C_j)$ which contains infinitely many points of the sequence $[P_i]$. Let $n_j$ be the smallest integer greater than $n_{j-1}$ such that $P_{n_j}$ lies in $D_j$. For every $j$, $D_j$ contains an arc $P_{n_j}P_{n_{j+1}}$.† Since for every $j$, the arc $P_{n_j}P_{n_{j+1}}$ lies interior to $C_j$ we see easily that the set $P + \sum_{j=1}^{\infty} P_{n_j}P_{n_{j+1}}$ is closed.

B. The condition is sufficient. If $N$ is not a continuous curve there exist‡ two concentric circles $K_1$ and $K_2$ and a countable infinity of continua $\overline{N}$, $N_1$, $N_2$, $N_3$, $\cdots$, such that (1) each of these continua belongs to $N$, contains a point on $K_1$ and a point on $K_2$ and is a subset of the set $H$ which is composed of $K_1+K_2+I$, $I$ denoting the annular domain between $K_1$ and $K_2$, (2) no two of these continua have a point in common and, indeed, no one of them except possibly $\overline{N}$ is a proper subset of any connected subset of $N \cdot H$.

* See the abstract of my paper, Concerning the arcs and domains of a continuous curve, this Bulletin, vol. 32 (1926), p. 37.
‡ See Report, p. 296.
(3) the set \( \overline{N} \) is the sequential limiting set of the sequence of sets \( N_1, N_2, N_3, \ldots \). For each \( i \), let \( A_i \) and \( B_i \) be points of \( K_1 \cdot N_i \) and \( K_2 \cdot N_i \) respectively. There exist arcs \( X_1 Y_1 A_i \) and \( X_2 Y_2 B_i \) of \( K_1 \) and \( K_2 \) and an increasing sequence of integers \( n_1, n_2, n_3, \ldots \), such that \( X_1 Y_1 A_i \) contains \( A_{n_i} \) for every \( i \) and in the order \( X_1 Y_1 A_{n_1} A_{n_2} \ldots A \) and \( X_2 Y_2 B_i \) contains \( B_{n_i} \) for every \( i \) and in the order \( X_2 Y_2 B_{n_1} B_{n_2} \ldots B \).

Let \( P \) denote a point of \( \overline{N} \) which lies in \( I \). There exists a circle \( C_1 \) with center at \( P \) such that \( C_1 \), together with its interior, lies in \( I \). Let \( r_1 \) be the radius of \( C_1 \). Since \( M \) is connected im kleinen at \( P \) there exists in any circle \( C_i \) a concentric circle \( \overline{C}_i \) such that every point of \( M \) within \( \overline{C}_i \) can be joined to \( P \) by an arc of \( M \) lying wholly within \( C_i \). Let \( N_{11} \equiv N_{n_{j1}} \), where \( j \) has the smallest value such that \( N_{n_j} \) contains a point \( Q_1 \) within \( \overline{C}_1 \). There exists an arc \( PQ_1 \) of \( M \) lying wholly in \( C_1 \). The arc \( PQ_1 \) from \( P \) to \( Q_1 \) contains a first point \( E_1 \) in common with \( N_{11} \) and the arc \( E_1 P \), a subset of \( Q_1 P \), has a first point \( F_1 \) in common with \( \overline{N} \). The set \( \{E_1 F_1\} \) contains a point \( P_1 \) of \( M - N \). Let \( C_2 \) be a circle with center at \( P \) and radius \( r_2 \equiv r_1/2 \) such that \( P_1 \) and \( N_{11} \) lie in the exterior of \( C_2 \). Let \( N_{12} \equiv N_{n_{j2}} \), where \( j \) has the smallest value such that \( N_{n_j} \) contains a point \( Q_2 \) within \( \overline{C}_2 \). Let us determine a point \( P_2 \) of \( M - N \) as above. Continue this process indefinitely each time taking \( C_i \) with center at \( P \) and radius \( r_i \equiv r_{i-1}/2 \) and such that \( P_{i-1} \) and \( N_{1i-1} \) lie outside \( C_i \). Thus we obtain an infinite sequence of points \( P_1, P_2, P_3, \ldots \), and continua \( N_{11}, N_{12}, N_{13}, \ldots \), such that (1) \( P_i \) belongs to \( M - N \) and lies interior to \( C_i \) and thus \( P \) is the sequential limit point of the sequence \([P_i]\), (2) \( \{E_i F_i\} \) contains \( P_i \), where \( C_i \) encloses \( E_i F_i \), and \( \{E_i F_i\} \) contains no point of \( N_{1i} + \overline{N} \).

Since \( M - N \) consists of only a finite number of maximal connected subsets one of these must contain infinitely many of the points \([P_i]\) say \( \overline{P}_1, \overline{P}_2, \overline{P}_3, \ldots \). For each \( i \), let \( D_i \) be the maximal connected subset of \( M + K_1 + K_2 - (\overline{N} + \overline{N}_{1i}) \).

* If \( AB \) is an arc from \( A \) to \( B \) then \( \{AB\} \) denotes \( AB - (A + B) \).
+K_1+K_2)\), which contains \(P_i\). We see easily that there exists an integer \(t_2\) such that \(\overline{P}_1\) does not lie in \(D_{t_2}\). Then any arc of \(M-N\) from \(\overline{P}_{i_1}\) to \(\overline{P}_{i_2}\) must contain a point of either \(K_1\) or \(K_2\). There exists an integer \(t_2>2\) such that \(D_{t_2}\) does not contain \(\overline{P}_{i_2}\). In general there exists an integer \(t_i>t_{i-1}\) such that \(D_{t_i}\) does not contain \(\overline{P}_{i_i}\) and thus any arc of \(M-N\) from \(\overline{P}_{i_{i-1}}\) to \(\overline{P}_{i_i}\) must contain a point of \(K_1\) or \(K_2\). Let \(P_i=\overline{P}_{i_i}\). Then if \(k_1, k_2, \ldots\) is an increasing sequence of positive integers, the set \(\overline{N}\) must contain a limit point of the set \(P_1+\sum_{i=1}^{\infty} P_{k_i} P_{k_{i+1}}\) which lies on \(K_1\) or \(K_2\) and thus the set cannot be closed. But this set is closed by hypothesis. Thus the condition is sufficient.

Theorem II. In order that a bounded plane continuum \(M\) be a continuous curve, it is necessary and sufficient that (1) for any given positive number \(\epsilon\) there are not more than a finite number of complementary domains of \(M\) of diameter greater than \(\epsilon\); (2) if \(P_1, P_2, P_3, \ldots\) is any sequence of distinct points of a complementary domain \(D\) of \(M\) which has a sequential limit point \(P\), then there exists an increasing sequence of positive integers, \(n_1, n_2, n_3, \ldots\), and a sequence of arcs of \(D\), \(P_{n_1} P_{n_2}, P_{n_2} P_{n_3}, P_{n_3} P_{n_4}, \ldots\), such that the set \(P_1+\sum_{i=1}^{\infty} P_{n_i} P_{n_{i+1}}\) is closed.

Proof. The necessity of condition (1) has been proved by Schoenflies. The necessity of condition (2) can be proved exactly as in Theorem I since no property of the continuous curve \(M\) was used that is not also a property of the entire space. The sufficiency of the conditions is proved as in Theorem I except that the fact that some one complementary domain of \(M\) contains infinitely many of the points \(P_1, P_2, P_3, \ldots\), which are chosen in the course of the argument, follows from condition (1) rather than the condition \(M-N\) consists of a finite number of maximal connected subsets.

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* If \(\overline{P}_i=P_i\), then \(N_{i+1}\) denotes \(N_i\).
† For the proof that such an arc exists, see R. L. Moore, Concerning continuous curves in the plane, loc. cit.
‡ See Report, pp. 290, 291.
THEOREM III. If $M$ is a plane continuous curve then $M$ cannot contain, for any positive number $\epsilon$, an infinite number of mutually exclusive continua $M_1, M_2, M_3, \cdots$, such that

(1) the diameter of each set $M_i$ is greater than $\epsilon$, (2) $M - M_i$ is closed except for a set $K_i$ and if $\eta$ is any positive number there exists an integer $n_\eta$ so that if $i > n_\eta$ then $K_i$ can be enclosed in two circles each of radius less than $\eta$.

PROOF. Suppose that there exists a positive number $\epsilon$ and a continuous curve $M$ such that $M$ contains an infinite number of continua $M_1, M_2, M_3, \cdots$, which satisfy restrictions (1) and (2) of the theorem. From condition (2) it follows that we may divide each set $K_i$ into two subsets $K_{1i}$ and $K_{2i}$ such that

$$\lim_{i \to \infty} d(K_{1i}) = 0 \text{ and } \lim_{i \to \infty} d(K_{2i}) = 0.$$  

For each $i$ and $j$ ($i = 1, 2, 3, \cdots, j = 1, 2$) let $A_{ij}$ be a point of $K_{ij}$, unless $K_{ij}$ is vacuous. For no value of $i$ can both $K_{1i}$ and $K_{2i}$ be vacuous. Several cases arise here according to the existence or non-existence of the various points $A_{ij}$ but we can see easily that there exist a point $A$ or two points $A$ and $B$ and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that either (1) $K_{1n_i}$ is vacuous for each $i$, and $A$ is the sequential limit point of $[A_{2n_i}]$, (2) $K_{2n_i}$ is vacuous for each $i$ and $A$ is the sequential limit point of $[A_{1n_i}]$, (3) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and $A$ is the sequential limit point of each sequence, or (4) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and $A$ and $B$ are the sequential limit points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ respectively ($A \neq B$). For cases (1), (2) and (3), let $t = \epsilon$; for case (4) let $t = d(A, B)$. By condition (2) of the hypothesis of the theorem, there exists an integer $k_1$ so that if $i > k_1$ then

$$d(K_{1n_i}) < t/12 \quad \text{and} \quad d(K_{2n_i}) < t/12.$$  

* If $K$ is a set of points the notation $d(K)$ denotes the diameter of $K$. If $A$ and $B$ are two points the notation $d(A, B)$ denotes the distance from $A$ to $B$. 
Also there exists an integer $k_2$ so that if $i > k_2$ then either

Case (1) $d(A_{2ni}, A) < t/12$,
or

Case (2) $d(A_{1ni}, A) < t/12$,
or

Case (3) $d(A_{1ni}, A) < t/12$ and $d(A_{2ni}, A) < t/12$,
or

Case (4) $d(A_{1ni}, A) < t/12$ and $d(A_{2ni}, B) < t/12$.

In any case if $k_3 = k_1 + k_2$ and $i > k_3$ then the circle $C_1$ with center at $A$ and radius $t/6$, or the circles $C_1$ and $C_2$ with centers at $A$ and $B$ and radii $t/6$, enclose every point of $K_{ni}$. For every $i > k_3$, $M_{ni}$ contains a point $p_i$ such that $d(A, p_i) > t/3$ and $d(B, p_i) > t/3$ (if $B$ exists). The sequence $M_{n_1}, M_{n_2}, M_{n_3}, \ldots$, contains a subsequence $M_1, M_2, M_3, \ldots$, such that (1) for every $i$, if $M_i = M_{nj}$ then $j > k_3$, (2) for every $i$, if $M_i = M_{nj}$ and $M_{i+1} = M_{nm}$ then $j < m$, (3) the points $p_1, p_2, p_3, \ldots$ have a sequential limit point $P$. It follows that $M$ contains $P$, that $d(P, A) \geq t/3$ and $d(P, B) \geq t/3$ (if $B$ exists) and that if $C_3$ is a circle of radius $t/6$ with $P$ as a center then no point of any set $K_i$ is within $C_3$. As $M$ is connected im kleinen at $P$ the circle $C_3$ encloses a concentric circle $C_4$ such that every point of $M$ within $C_4$ can be joined to $P$ by an arc of $M$ which lies entirely in $C_3$. Let $ar{p}_s$ be the first point of the sequence $[\bar{p}_1]$ within the circle $C_4$. There exists an arc $\alpha$ from $\bar{p}_s$ to $P$ which lies wholly in $C_3$. Let $\alpha_1 = M_s \cdot \alpha$ and $\alpha_2 = \alpha - \alpha_1$. As $\alpha$ is connected one of these sets must contain a limit point of the other. The set $M_s$, and consequently $\alpha_1$, is closed. Then $\alpha_1$ must contain a limit point $q$ of $\alpha_2$. As $M_s$ contains $\alpha_1$ and $M - M_s$ contains $\alpha_2$, by definition $q$ must belong to $K_s$. But no point of $K_s$ is within $C_3$ while $\alpha$ is entirely within $C_3$. Thus the supposition that $M$ contains an infinite set of this type has led to a contradiction.

The preceding theorem implies as an immediate corollary the following rather useful result.

* If $M_i = M_{nj}$, then $P_i$ denotes $p_{ni}$.
THEOREM IV.* If $M$ is a plane continuous curve then $M$ cannot contain, for any positive number $\epsilon$, an infinite number of arcs of diameter greater than $\epsilon$ which are mutually exclusive except possibly for common end-points and such that if $\alpha$ is any one of this set of arcs then $M - \{\alpha\}$ is closed.

That Theorem I no longer remains true when the condition that "$M - N$ consists of a finite number of maximal connected subsets" is removed, even with the addition of the condition that "for any positive number $\epsilon, M - N$ contains only a finite number of maximal connected subsets of diameter greater than $\epsilon," is shown by the following example. The modified conditions are necessary but not sufficient.

Let $N$ denote the set of points consisting of the intervals from $(1, 0)$ to $(0, 0)$ and from $(0, 1)$ to $(0, 0)$ together with the intervals from $(1, 1/i)$ to $(0, 1/i)$ for every positive integer $i$. Let $M$ be the set of points consisting of $N$ together with the intervals from $(j/i, 1/i)$ to $(j/i, 0)$ for every positive integer $j < i$ and for every positive integer $i$. The modified conditions are then satisfied, but $N$ is not a continuous curve.

Theorem III gives a necessary condition that a bounded continuum be a continuous curve. The following example shows that this condition is not sufficient:† Let $M$ be an indecomposable continuum of diameter $\geq 2\epsilon$ and let $\eta < \epsilon/10$. Now suppose that $M_1, M_2, M_3, \ldots$ is any sequence of mutually exclusive subcontinua of $M$ of diameter greater than $\epsilon$. As each set $M_i$ is a proper subcontinuum of an indecomposable continuum it is a continuum of condensation of $M$.‡ Thus for each $i$, $K_i \equiv M_i$. Then no matter how large $i$ is, $K_i$ cannot be enclosed in two circles each of radius less than $\eta$. Thus $M$ satisfies the condition but is not a continuous curve.

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* This theorem was presented to the Society October 31, 1925. I am indebted to Dr. H. M. Gehman for the suggestion that this theorem might be generalized. The resulting study led to Theorem III of this paper.

† This example is due to Professor J. R. Kline.