ON A GENERALIZATION OF THE SECULAR EQUATION*

BY JAMES PIERPONT

1 Introduction. The equation we wish to consider is

\[ H_n(x) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0. \]

Here \( a_{ij} = a_{ji} \), when \( i \neq j \); while
\[ a_{ii} = \alpha_{ii} - x, \quad \text{for } i = 1, 2, \ldots, r, \]
\[ = \alpha_{ii} + x, \quad \text{for } i = r + 1, r + 2, \ldots, n. \]

The \( a_{ij} \) and \( \alpha_{ii} \) are real, and \( H(0) \neq 0 \). If \( r = n \), (1) is the secular equation which it will be convenient to denote by \( L_n(x) = 0 \). When \( r = n - 1 \) the equation (1) plays a fundamental role in classifying quadric surfaces in \( n \)-way hyperbolic space. Let us set \( n - r = s \) and call \( \sigma = |r - s| \) the signature of (1). We have then the

**THEOREM I.** The number of real roots of \( H_n(x) = 0 \), counting their multiplicity, is not less than its signature.

This is a corollary of a theorem to which F. Klein calls especial attention (Mathematische Annalen, vol. 23 (1884), p. 562). The proof there given rests on the theory of elementary divisors.† We give here a very simple proof which is a modification of H. Weber’s proof that the roots of the secular equation \( L_n(x) = 0 \) are all real.‡ Weber’s proof as we shall see, is complicated by his belief that it is necessary to show that

\[ L_n'(x) = - \sum_i \frac{\partial L_n}{\partial a_{ii}}, \quad (i = 1, 2, \ldots, n). \]

* Presented to the Society, December 29, 1926.
2. Proof of the Theorem. We turn now to the proof of the above Theorem I. Consider the sequence

\[ H_n, H_{n-1}, H_{n-2}, \ldots, H_1, H_0 = 1, \]

where \( H_k \) is the determinant of degree \( k \) in \( x \) obtained by deleting the last \( n-k \) rows and columns of (1). For the moment we suppose that no two of the \( H \)'s vanish for the same \( x \). They are connected by the relations

\[
\begin{align*}
H_n H_{n-2} &= H_{n-1} \phi_{n-1} - \psi_{n-1}^2, \\
H_{n-1} H_{n-3} &= H_{n-2} \phi_{n-2} - \psi_{n-2}^2, \\
& \quad \vdots \\
H_2 H_0 &= H_1 \phi_1 - \psi_1^2,
\end{align*}
\]

where \( \phi, \psi \) are polynomials in \( x \).

Merely for completeness let us show how these relations are obtained, the first for example. Let \( A_{ij} \) be the minor of \( a_{ij} \) in (1); set \( \nu = n-1 \). Then

\[
B = \begin{vmatrix} A_{\nu\nu} & A_{\nu n} \\ A_{n\nu} & A_{nn} \end{vmatrix} = H_{n-1} A_{\nu\nu} - A_{n\nu}^2.
\]

But \( H_n \cdot B = H_{n-2} \cdot H_n^2 \). Hence, if \( H_n \neq 0 \), \( B = H_{n-2} \cdot H_n \). This with (4) gives

\[ H_n H_{n-2} = H_{n-1} \phi_{n-1} - \psi_{n-1}^2. \]

This relation holds also if \( H_n = 0 \), as continuity considerations show.*

The equations (3) show that when \( H_k = 0 \), \( H_{k+1}, H_{k-1} \) have opposite signs.

We now consider the signs of the sequence (2). Suppose \( H_n = 0 \) for \( x = a \). Then in a sufficiently small interval \( \delta \) about \( x = a \), \( H_n \) changes its sign, while none of the other terms in (2) do. Thus as \( x \) passes through \( \delta \) the sequence (2) gains or loses one variation of sign. On the other hand when \( x \) passes through a root of \( H_{n-1}, H_{n-2}, \ldots \) no variation is gained or lost as (3) show. For \( x = \pm \infty \) there are \( r \)

* See Weber, loc. cit., p. 113; or Kowalewski, *Determinantentheorie*, p. 83.
variations of sign in (2); for \( x = -\infty \) there are \( s \), thus
\[ H_n(x) = 0 \] has at least \( \sigma \) real roots.

We now consider the general case that the sequence (2) has common roots. With Weber we may dispose of this case as follows. Suppose e.g. that \( H_k, H_{k-1} \) have common roots. We vary the terms \( a_{ij} \) of \( H_k \) not in \( H_{k-1} \) by small amounts numerically less than some \( \eta \), so that \( H_k, H_{k-1} \) do not have common roots.

In this way we may replace (2) by another sequence

\[
K_n, K_{n-1}, K_{n-2}, \ldots, K_1, K_0 = 1
\]

no two of which have a common root. The roots of \( K_n = 0 \) differ from those of \( H_n = 0 \) by an amount as small as we please, for sufficiently small \( \eta \), moreover the signs of corresponding elements of the sequences (2), (5) are the same for an \( x \) for which no element of (2) vanishes. As Theorem I holds for (5), it must hold for (2).

**Theorem II.** _The roots of the secular equations are all real._

For in this equation \( s = 0 \); hence \( \sigma = n \).

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**A GENERALIZED TWO-DIMENSIONAL POTENTIAL PROBLEM**

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It may be shown that the solution of the problem of electromagnetic wave propagation along a system of straight parallel conductors can be reduced to the solution* of two subsidiary problems: (1) a well known problem in two-dimensional potential theory; and (2) a generalization of the two-dimensional potential problem which is believed to be novel. The generalized problem is believed to possess suffi-

* Subject to certain restrictions to be discussed in a forthcoming paper.