ON HILBERT'S THIRTEENTH PARIS PROBLEM*

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At the Paris Congress in 1900, Hilbert† presented for proof the proposition that the function $f$ of the three variables $x$, $y$, and $z$ satisfying the equation

\[(1) \quad f' + xf^3 + yf^2 + zf + 1 = 0\]

cannot be represented by the use of a finite number of continuous functions of not more than two arguments. In this note a small part of this problem is considered. We shall prove that the function $f$ cannot have the form $F[a(x, y), P(y, z)]$, where $F(\alpha, \beta)$, $\alpha(x, y)$ and $\beta(y, z)$ are analytic functions.

Before proceeding to the proof it is necessary to notice certain properties of the partial derivatives $f_x$, $f_y$, and $f_z$. They satisfy the identities

\[(2) \quad Uf_x = f^3, \quad Uf_y = f^2, \quad Uf_z = f,\]

where $U = -(7f^3 + 3xf^2 + 2zf + z) \neq 0$. For finite values of $x$, $y$, and $z$, $f$ is finite and does not vanish. Hence $U$ is finite and therefore the first partial derivatives cannot vanish.

In the proof we assume that

\[(3) \quad f(x, y, z) = F[\alpha(x, y), \beta(y, z)],\]

where $F(\alpha, \beta)$, $\alpha(x, y)$, and $\beta(y, z)$ are analytic functions. Since $f_z \neq 0$ and $f_x \neq 0$, $\alpha_x \neq 0$ and $\beta_z \neq 0$ for finite values of $x$, $y$, and $z$. The Jacobian condition for functional dependence

\[\begin{vmatrix}
  f_z & f_y & f_z \\
  \alpha_x & \alpha_y & 0 \\
  0 & \beta_y & \beta_z \\
\end{vmatrix} = 0\]

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can then be written in the form

\[ A(x, y)f_x + f_y + C(y, z)f_z = 0, \]

where \( A = -\alpha_y/\alpha_x \) and \( C = -\beta_y/\beta_z \). Multiplying by \( U \), making use of (2), and subsequently dividing by \( f \), we get

\[ A(x, y)f_x + f + C(y, z) = 0. \]

It is now easy to show that \( A \) is linear in \( x \). Differentiating (4) with respect to \( x \) and separately with respect to \( z \) we find by the use of (2) that \( A_z = C_z \). Since \( C \) does not contain \( x \), \( C_z \) does not contain \( x \) and hence \( A_x \) is a function of \( y \) alone.

We shall now show that \( A \) does not contain \( x \) at all. The eliminant with respect to \( f \) between (1) and (4) is an identity in \( x \) that has for its term in the highest power of \( x \) the term contributed by the expansion of \( A \). This term must vanish identically and hence the coefficient of \( x \) in \( A \) must vanish identically. But if \( A \) does not contain \( x \) we have by (4) \( f_x = 0 \). This is impossible and the assumption that \( f \) satisfies (3) leads to a contradiction.

Similarly it can be shown that \( f \) cannot have either the form \( F[\alpha(x, y), \beta(x, z)] \) or the form \( F[\alpha(x, z), \beta(y, z)] \); all functions being assumed analytic.