APPELL ON TENSOR CALCULUS

Traité de Mécanique Rationnelle. Vol. V: *Eléments de Calcul Tensoriel, Applications Géométrique et Mécanique.* By Paul Appell with the collaboration of René Thiry. Paris, Gauthier-Villars, 1926. vi+198pp.

This work of Appell on the absolute differential calculus and its applications forms the 5th volume of his *Traité de Mécanique Rationnelle* and is intended as the first part of a treatment of the mechanics of the theory of relativity. Yet it would seem that the present is no time for the appearance of such a text. The uncertain position of the general theory of relativity as shown by the unsuccessful attempts of Einstein and others to develop a field theory of electromagnetic phenomena, together with the new views, beginning with Heisenberg and ending with Schrödinger, which have lately been advanced, show that we do not yet stand on solid ground. It may be necessary to make substantial changes in the mathematical foundations of relativity theory. As a consequence it is better to regard this last book of Appell merely as an elementary exposition of some work in pure mathematics without regard to applications.

One feature of the book which deserves considerable adverse criticism is the lack of any *real* application to mechanics. A few of the simpler equations of mechanics have been written in the tensor notation on the basis of the three-dimensional euclidean geometry. But no applications to mechanics of the more general mathematical developments have been made.

The first chapter is devoted to certain fundamental theorems on linear and quadratic forms, and as such is introductory to the tensor calculus proper; an especially small type of printing is employed. A treatment of these fundamental theorems is usually omitted by writers on this subject. I regard this chapter as a valuable addition to the book.

Owing to the all too many books which have appeared in recent years on the theory of relativity, as well as the increasing number of mathematical papers written on the basis of the tensor calculus, there does not remain much room for originality in the second chapter which is devoted to the elements of this subject. However, on account of the importance of this chapter for the later developments, it may be well to consider it in some detail. Going out from the two fundamental requirements governing the equations of transformation of *a system of functions* (f), see §20, the system of functions known as a tensor is introduced as a simple system satisfying the fundamental requirements. After this the rules of tensor algebra are laid down, the fundamental quadratic differential form $g_{\alpha\beta}dx^\alpha dx^\beta$ is produced, and the differential equations of geodesics are developed in the ordinary manner. Then follows a cumbersome treatment of covariant differentiation of a tensor on the basis of the artificial and foreign idea of a comparison of tensors at different points of the manifold. In reality, covariant differentiation of a tensor has its natural origin in the determination of the conditions of integrability of the equations of transformation of the components of the tensor itself and those of the fundamental tensor $g_{\alpha\beta}$. Next the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ is deduced together with certain identities satisfied by it. The chapter closes with the derivation of certain well known formulas.

Much of the work in this chapter, and in the rest of the book for that matter, could be simplified very much by the systematic use of normal coordinates, and in particular by a consideration of the relationship of the process of covariant differentiation to this system of coordinates. In this connection there arise the "extensions" of a tensor which in general are distinct from its covariant derivatives and which have important theoretical applications. For example, to give an almost trivial case, the extensions of the fundamental tensor $g_{\alpha\beta}$ may be used to derive the complete set of identities of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ from which *all* other identities satisfied by this tensor can be deduced by algebraic processes. Appell has nothing to say as to whether the identities (27) p. 58 constitute all the identities satisfied by the tensor $R_{\alpha\beta\gamma\delta}$, and I take it that he is not quite sure of himself on this point since the identities (27) are not all algebraically independent among themselves.

Now a remark regarding notations! It is somewhat helpful in calculation to write the Christoffel three-index symbols of the second kind in the inverted form $\left\{ {}_{\alpha}{}^{i}{}_{\beta} \right\}$ or to denote them by a symbol such as $\Gamma_{\alpha}{}^{i}{}_{\beta}$ in order that the summation of indices may be in conformity with the general rule (see §47, p. 47). For the purpose of emphasizing the operation of covariant differentiation Appell, following Galbrun, denotes, for example, by $\Delta_k A_r$ the covariant derivative of the vector A_r (see §47, p. 50). I do not regard this as a desirable notation. The objections to the cumbersome symbol $\sum$ to denote a summation likewise apply to the Δ notation, and in operations involving repeated covariant differentiation, where a number of Δ's appear before a tensor, these amount to nothing more than a succession of stumbling blocks in the way of ready calculation. What is needed is a notation for covariant differentiation such as $A_{r/k}$ or $A_{r,k}$ which will bring this operation into evidence but in as slight a manner as possible.

Beginning with the third chapter, the book will probably take on an added interest for most readers. The euclidean space of three dimensions is discussed with reference to general curvilinear coordinates. The formulas for area and volume are brought in, and an interesting geometrical interpretation of the covariant and contravariant components of a tensor of the first order is given. To this chapter also belong the mechanical formulas which I have already mentioned.

The fourth and fifth chapters deal with the euclidean and Riemannian spaces of n dimensions respectively. The fourth chapter contains the generalization of the geometrical portion of the third, and in addition a consideration of subspaces of the euclidean space, parallel displacement of vectors, etc. The proof of the sufficiency of the condition $R_{\alpha\beta\gamma\delta}=0$ for the space to be flat is very straightforward analytically and as good as I have ever come across. However, it is not quite clear just what is meant by $\Delta_k a^i$ in the generalization of the first formulas of Serret-Frenet in §85, since a^i is a function of a single parameter and not of the independent variables (x), as is necessary for the process of covariant differentiation. Certainly the fifth chapter containing the treatment of Riemann space by the method of Levi-Civita, which consists in regarding this space as a subspace of a euclidean space of a larger number of dimensions, is one of the most interesting in the book. But from the standpoint of physical application, it is the intrinsic point of view essentially which is demanded.

For the most part, the sixth chapter is devoted to the presentation of the affine and metric manifolds of H. Weyl. The method of development is not that used by Weyl in *Raum, Zeit, Materie*, but one which has been used so extensively by Cartan in which a plane manifold is associated with each point of the given continuum. A portion of the chapter is also given to the spaces of torsion of Cartan and there is a short paragraph on the geometry of Eddington. This latter, however, gives a very incomplete account of the views of Eddington and it would seem that a theory which has had such a stimulating influence in this field could have more space allotted to its description.

My objection to the work of Cartan is that it consists in a large measure of a mere superabundance of geometrical terminology; and this applies even after its analytical structure has been improved by use of the tensor calculus. Its lack of content is evident when once the pure invariant theoretic viewpoint is adopted.

Finally, there is a chapter in small type on non-euclidean geometries as treated by Cayley, which might, perhaps, have been made more interesting and colorful by more verbal detail.

I believe, however, that the book will be very helpful to students wishing to gain a first insight into this mathematical subject, in spite of the criticisms which I have made, since as a whole it is written very clearly. It should prove quite satisfactory for use as an elementary treatise as intended, in fact, by its author. But this is no more than might be expected, for the genius of Appell as a clear and lucid expositor is recognized by everyone.

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