CONCERNING THE BOUNDARIES OF DOMAINS
OF A CONTINUOUS CURVE*

BY W. L. AYRES

We shall consider a space $M$ consisting of all the points of a plane continuous curve $M$, and all point sets mentioned are assumed to be subsets of $M$. A connected set of points $D$ of $M$ is said to be an $M$-domain if $M - D$ is closed. The set of all limit points of $D$ which do not belong to $D$ is called the $M$-boundary of $D$. If $D$ is an $M$-domain, $D'$ denotes the $M$-boundary of $D$, and $\bar{D}$ denotes the set $D + D'$. The $M$-boundary of an $M$-domain $D$ is closed but not necessarily connected, even if $D$ is simply connected, as we may easily show by examples. If $N$ is a continuum, a maximal connected subset of $M - N$ is called a complementary $M$-domain of $N$.

**Theorem I.** Every closed and connected subset of the $M$-boundary of a complementary $M$-domain of a continuous curve $N$ is a continuous curve.

**Proof.** Let $K$ denote a closed and connected subset of the $M$-boundary of a complementary $M$-domain $D$ of $N$. Suppose $K$ is not connected im kleinen. Then there exist two concentric circles $C_1$ and $C_2$ (let $r_1$ denote the radius of $C_1$ and let $r = r_1 - r_2 > 0$) and an infinite sequence of subcon-
tinua of $K, K_a, K_1, K_2, K_3, \ldots$, such that (1) each of these continuas $K_a (a = \infty, 1, 2, \ldots)$ contains a point $a_\alpha$ on $C_1$ and a point $b_\alpha$ on $C_2$, but no point exterior to $C_1$ or interior to $C_2$, (2) no two of these continuas have a point in common and no one of them, except possibly $K_\infty$, is a proper subset of any connected subset of $K$ which contains no point without $C_1$ or within $C_2$, (3) the set $K_\infty$ is the sequential limiting set of the sequence of continuas $K_1, K_2, K_3, \ldots$. For any $i$, let $K'_i = K_\infty + K_{i+1} + K_{i+2} + \cdots$. Let $d_1$ be the smaller of the numbers $\frac{1}{4} r$ and $\frac{1}{2} d(K_1, K_1^*)$.

Let $R_1$ denote the set of all points $[P]$ of $N$ such that the distance from $P$ to some point of $K_1$ is less than $d_1$. The set $R_1$ is an open subset of $N$ and hence $R_1$ contains an arc from $a_1$ to $b_1$. This arc contains a subarc $x_1y_1$ such that $x_1$ is on $G$ and $y_1$ is on $G$ and every other point of $x_1y_1$ is between $C_1$ and $C_2$.

For each $n > 1$, let $d_n$ be the smallest of the numbers $d_{n-1}, r2^{-n-1}$, and $\frac{1}{2} d(K_n, K_n^* + x_1y_1 + x_2y_2 + \ldots + x_{n-1}y_{n-1})$. Let $R_n$ denote the set of all points $[P]$ of $N$ such that there is some point of $K_n$ whose distance from $P$ is less than $d_n$. The set $R_n$ contains an arc from $a_n$ to $b_n$ and this arc contains a subarc $x_ny_n$ such that $x_n$ is on $C_2$, $y_n$ is on $C_1$, and every other point of $x_ny_n$ is between $C_1$ and $C_2$.

There exists an increasing sequence of positive integers $n_1, n_2, n_3, \ldots$, such that (1) $C_2$ contains three points $X_1, X_2, X_3$ such that every point $x_{n_i}$ lies on the arc $X_1X_2X_3$ of $C_2$ and in the order $X_1X_2x_{n_2}x_{n_3} \cdots X_3$, (2) $C_1$ contains three points $Y_1, Y_2, Y_3$ such that every point $y_{n_i}$ lies on the arc $Y_1Y_2Y_3$ of $C_1$ and in the order $Y_1Y_2y_{n_2}y_{n_3} \cdots Y_3$, (3) $X_3$ is the sequential limit point of $[x_{n_3}]$ and $Y_3$ is the sequential limit point of $[y_{n_3}]$. Clearly $K_\infty$ contains $X_3$ and $Y_3$.

† If $A$ and $B$ are sets of points, the symbol $d(A, B)$ denotes the lower bound of all numbers $d(x, y)$, where $x$ is a point of $A$, $y$ is a point of $B$, and $d(x, y)$ is the distance from $x$ to $y$.

The set $K_\omega$ contains points $X$ and $Y$ on the circles which are concentric with $C_1$ and with radii $r_2 + r/10$ and $r_1 - r/10$ respectively. Let $\eta$ be the smaller of the two numbers $r/10$ and $r_2$. Since $N$ is connected im kleinen, there exists a positive number $\delta_\eta$ such that any point of $N$ within a distance $\delta_\eta$ of $X$ or $Y$ may be joined to $X$ or $Y$, as the case may be, by an arc of $N$ every point of which is within a distance $\eta$ of $X$ or $Y$. Let $n_1$ be the smallest integer such that $x_{n_1}y_{n_1}$ contains two points $P$ and $Q$ such that $d(P, X) < \delta_\eta$ and $d(Q, Y) < \delta_\eta$. Then $N$ contains arcs $PX$ and $QY$, every point of which is within a distance $\eta$ of $X$ and $Y$ respectively. Let order be defined on these arcs as from $P$ to $X$ and from $Q$ to $Y$. The arcs $PX$ and $QY$ have points in common with every arc $x_{n_i}y_{n_i}$ for $i \geq 8$.

Let $V_{10}$ and $V_{20}$ be the last points the arcs $PX$ and $QY$ have in common with $x_{n_i}y_{n_i}$. Let $U_{11}$ and $U_{21}$ be the first points and $V_{11}$ and $V_{21}$ be the last points the subarcs $V_{10}X$ and $V_{20}Y$ have in common with $x_{n+1}y_{n+1}$. Let $U_{12}$ and $U_{22}$ be the first points the subarcs $V_{11}X$ and $V_{21}Y$ have in common with $x_{n+2}y_{n+2}$. The set $J$, composed of the arcs $V_{10}V_{20}$ of $x_{n_i}y_{n_i}$, $U_{11}V_{11}$ and $U_{21}V_{21}$ of $x_{n+1}y_{n+1}$, $U_{12}U_{22}$ of $x_{n+2}y_{n+2}$, $V_{10}U_{11}$ and $V_{11}U_{12}$ of $PX$, $V_{20}U_{21}$ and $V_{21}U_{22}$ of $QY$, is a simple closed curve. The subarc of $x_{n+1}y_{n+1}$ lying within $J$ and the arc $x_{n+2}y_{n+2}$ have points $p_1$ and $p_3$, respectively, in common with the circle concentric with $C_1$ and with radius $r_2 + \frac{3}{2}r$. The point $p_1$ is interior to $J$ and within a distance $d_{n+1}$ of some point of $K$ and this point is a limit point of $D$. Thus $D$ contains a point in the interior of $J$. Similarly $D$ contains a point within a distance $d_{n+2}$ of $p_3$ and thus in the exterior of $J$. Since $D$ is connected $D$ must contain a point of $J$. But as $J$ belongs to $N$ and $D$ to $M - N$, $D$ cannot contain a point of $J$. Thus the assumption that $K$ is not connected im kleinen has led to a contradiction.

**Theorem II.** If a maximal connected subset $K$ of the boundary of an $M$-domain is a continuous curve, every closed and connected subset of $K$ is a continuous curve.
DEFINITION. If $D$ is an $M$-domain and $P$ is a point of $M - \overline{D}$, the $M$-boundary of the maximal connected subset of $M - \overline{D}$ containing $P$ will be called the $M$-boundary of $D$ with respect to $P$. This is a generalization of the notion of outer boundary as defined by R. L. Moore. If $M$ is the entire plane, $D$ is bounded, and $P$ is a point of the maximal connected subset of $M - \overline{D}$ which is unbounded, then the $M$-boundary of $D$ with respect to $P$ is exactly the outer boundary of $D$ as defined by Moore.

THEOREM III.† If $D$ is an $M$-domain, $P$ is a point of $M - \overline{D}$, and $B$ is the $M$-boundary of $D$ with respect to $P$, then $B$ is the entire $M$-boundary of some $M$-domain which contains $D$.

PROOF. The entire set $D$ lies in the same maximal connected subset of $M - B$ and let $R$ denote this maximal connected subset. Evidently $R$ is an $M$-domain containing $D$. Suppose $Q$ is a point of the $M$-boundary of $R$. If $Q$ does not belong to $B$, $Q$ belongs to a maximal connected subset of $M - B$ which is different from $R$. Then $Q$ is not a limit point of $R$. Therefore every point of the $M$-boundary of $R$ is a point of $B$. Conversely every point of $B$ is an $M$-boundary point of $D$ and thus of $R$. Hence $B$ is identical with the $M$-boundary of $R$, an $M$-domain containing $D$.

THEOREM IV. If (1) $D$ is an $M$-domain and $P$ is a point of $M - \overline{D}$, (2) every maximal connected subset of $D'$ is a continuous curve, (3) the $M$-boundary of $D$ with respect to $P$, which we denote by $B$, is bounded, then every maximal connected subset of $B$ is either a point, a simple continuous arc or a simple closed curve.

PROOF. Let $R$ be the maximal connected subset of $M - \overline{D}$ containing $P$, and let $B_1$ be a maximal connected subset of $B$.

* Concerning continuous curves in the plane, loc. cit., p. 256.
† Compare R. L. Moore, Concerning continuous curves in the plane, loc. cit., Theorem 3, p. 258.
‡ See R. L. Moore, A characterization of a continuous curve, Fundamenta Mathematicae, vol. 7 (1925), Lemma 1, p. 302.
By Theorem II, $B_1$ is a continuous curve and $B_1$ is bounded by hypothesis. If $B_1$ consists of a single point, our theorem is proved. If $B_1$ consists of more than a single point then by a theorem due to Mazurkiewicz,* $B_1$ contains two points $x$ and $y$ which do not cut $B_1$. The continuous curve $B_1$ contains an arc $xzy$ from $x$ to $y$. If $B_1 = xzy$, our theorem is proved. If not, let $p$ be a point of $B_1$ which does not lie on $xzy$. By Theorem III, $B$ is the entire $M$-boundary of some $M$-domain $H$ which contains $D$. Clearly $R$ and $H$ are mutually exclusive and $B$ is the entire $M$-boundary of each. By a theorem due to Wilder,† if $p_1$ and $p_2$ are points of $R$ and $H$ respectively there exist arcs $p_1x$ and $p_1y$ which lie except for $x$ and $y$ wholly in $R$ and arcs $p_2x$ and $p_2y$ which lie except for $x$ and $y$ wholly in $H$. The sets $p_1x + p_1y$ and $p_2x + p_2y$ contain arcs $xuy$ and $xvy$ which lie wholly in $R$ and $H$ respectively except for the points $x$ and $y$.

Let $J_1$, $J_2$, $J_3$ be the simple closed curves formed of $xuy + xzy$, $xvy + xzy$, $xvy + xuy$ respectively and let $I_i$ and $E_i$ denote the interior and exterior of $J_i (i = 1, 2, 3)$. We have three cases to consider:

Case (1). Suppose $J_3 = I_1 + I_2 + <xzy>$.‡ Any point $q$ of $B - xzy$ lies either in $I_1$, $I_2$ or $E_3$. If $q$ lies in $I_1$, $I_1$ contains a point of $H$ since $q$ is a limit point of $H$. The exterior $E_1$ contains $<xvy>$ of $H$. But $H$ is connected and contains no point of $J_1$. Hence $I_1$ contains no point of $B - xzy$. Similarly $I_3$ contains no point of $B - xzy$. Then every point of $B - xzy$ lies in $E_3$. Since $x$ and $y$ are not cut-points of $B_1$, the continuous curve $B_1$ contains an arc $px$ which does not contain $y$ and an arc $py$ which does not contain $x$. The set $px + py$ contains an arc $xwy$ from $x$ to $y$. Since $p$ is in $E_3$ and no point of $J_3$ except $x$ and $y$ is a point of $B_1$, the set $<xwy>$ lies

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* Un théorème sur les lignes de Jordan, Fundamenta Mathematicae vol. 2 (1921), pp. 119—130.
† Loc. cit., Theorem 1, p. 342.
‡ If $xzy$ denotes a simple continuous arc with end-points $x$ and $y$ $<xzy>$ denotes $xzy - (x + y)$. 
entirely in $E_3$, and thus the arcs $xwy$ and $xzy$ have only $x$
and $y$ in common. Let $J_4$ be the simple closed curve $xzy + xwy$.
We will show that $B - J_4$ is vacuous and thus prove $B = B_1 = J_4$.
Suppose $B - J_4$ contains a point $q_1$. If $q_1$ lies in the interior
of $J_4$ both $R$ and $H$ have points in the interior of $J_4$ since
$q_1$ is a limit point of both domains. One of the two sets
$<xuy>$ or $<xvy>$, say $<xuy>$, lies entirely in the exterior
of $J_4$. Then $R$ contains points interior and exterior to $J_4$
but contains no point of $J_4$, which is impossible. If $q_1$ lies
in the exterior of $J_4$, both $R$ and $H$ contain points in the
exterior of $J_4$, and one of them contains points in the in-
terior, which is impossible. Therefore $B - J_4$ is vacuous,
which proves the theorem for this case.

*Case (2).* Suppose $I_2 = I_1 + I_3 + <xuy>$.
*Case (3).* Suppose $I_1 = I_2 + I_3 + <xvy>$.

In Cases (2) and (3), it may be proved by methods similar
to those of Case (1) that $B = B_1$ and $B_1$ is a simple closed
curve. Therefore $B_1$ is either a point, a simple continuous arc,
or a simple closed curve.

In proving Theorem IV we have obtained this result:

**Theorem V.** Under the hypothesis of Theorem IV, if any
maximal connected subset $J$ of the $M$-boundary of $D$
with respect to $P$ is a simple closed curve, then $J$ is the entire $M$-boundary
of $D$ with respect to $P$.

**Theorem VI.** If $D$ is an $M$-domain, $P$ is a point of $M - D$,
$R$ is the maximal connected subset of $M - D$ containing $P$, and
$Q$ is a point of the maximal connected subset of $M - \bar{R}$
which contains $D$, then $R'$ is the $M$-boundary of $R$ with respect to $Q$.

**Proof.** Let $H$ denote the maximal connected subset of
$M - \bar{R}$ which contains $D$. By definition $H'$ is the $M$-boundary
of $R$ with respect to $Q$. The set $H'$ is a subset of $R'$ since the
$M$-boundary of a domain with respect to a point is always a
subset of the $M$-boundary of the domain. By definition, $R'$
is the $M$-boundary of $D$ with respect to $P$. Then every point
of $R'$ is a limit point of $D$ and thus of $H$. As $H$ is a subset of
A THEOREM ON CONNECTED POINT SETS

BY C. KURATOWSKI AND C. ZARANKIEWICZ

1. Introduction. The purpose of this paper is to prove the following theorem.

THEOREM. If $S$ is a connected point set and $Z$ is the set of all points such that $S - p$ is neither connected nor the sum of two connected sets, then $Z$ is finite or countable.

2. Lemma. If $S$, $P$, and $Q$ are three non-vacuous connected sets (or points), and if

(1) $P + Q \subset S$,
(2) $P \cdot Q = 0$,
(3) $A \subset S - P$,
(4) $B \subset S - Q$,
(5) $A \cdot Q = 0$,
(6) $B \cdot P = 0$,
(7) $A$ and $S - P - A$ are mutually separated,
(8) $B$ and $S - Q - B$ are mutually separated,

then $A \cdot B = 0$.

Proof. By (1) and (3), $A + P \subset S$. Hence, by (2) and (5), $A + P = (A + P) \cdot (S - Q)$. By (4), $S - Q = B + (S - Q - B)$. Therefore

(9) $A + P = (A + P) \cdot B + (A + P) \cdot (S - Q - B)$.

It follows from (2) and (6) that $P = P - Q - B \subset (A + P)$ $(S - Q - B)$. Since $P \neq 0$, we have

(10) $(A + P) \cdot (S - Q - B) \neq 0$.

Now, by (8), the sets $(A + P) \cdot B$ and $(A + P) \cdot (S - Q - B)$ are mutually separated. On the other hand, by virtue of a