ON THE MAPPING OF THE SEXTUPLES OF THE SYMMETRIC SUBSTITUTION GROUP \( G_6 \)
IN A PLANE UPON A QUADRIC

BY ARNOLD EMCH

1. Introduction. The six permutations of three elements \( x_1, x_2, x_3 \) considered as projective coordinates in a plane determine an involution of sextuples of points which may be mapped on a rational surface.† I shall show that in case of the involution thus defined the map is a quadric whose relation with the plane, established with sufficient details, will lead to some interesting geometric applications. The map of every configuration on the quadric will be a configuration in the plane, invariant under the \( G_6 \), whose geometric properties have been investigated before.‡

2. Mapping of the \( G_6 \). Let \( \phi_1, \phi_2, \phi_3 \) represent the elementary symmetric functions \( \phi_1 = x_1 + x_2 + x_3, \phi_2 = x_2 x_3 + x_3 x_1 + x_1 x_2, \phi_3 = x_1 x_2 x_3 \), so that the general symmetric function of degree four has the form

\[
y_i = a_i \phi_i^4 + b_i \phi_i^3 \phi_2 + c_i \phi_i \phi_3 + d_i \phi_2^2,
\]

depending upon three effective constants. Hence, there are four linearly independent functions \( y_i; i = 1, 2, 3, 4 \), which we may set proportional to the four projective coordinates of a point in a space \( S_3 \). Thus to every point in \( (x) \), and consequently to every sextuple \( I_6 \{ (x_1 x_2 x_3), (x_1 x_3 x_2), (x_2 x_1 x_3), (x_3 x_1 x_2), (x_2 x_3 x_1) \} \), corresponds in \( S_3 \) a point \( (y) \), and as the system of sextuples is a continuous \( \infty^2 \) manifold, the locus of such points \( (y) \) must be a surface, which will be proved to be a quadric cone \( Q \). For the \( y_i \)'s we may

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evidently choose any four linearly independent symmetric quartics. For every choice of four such functions we obtain a certain surface \( Q \). But all these surfaces are obviously collinearly related. The simplest choice is

\[
\begin{align*}
\rho y_1 &= x_1^4 + x_2^4 + x_3^4 = \phi_1^4 - 4\phi_2^2\phi_3 + 4\phi_1\phi_3 + 2\phi_3^2, \\
\rho y_2 &= x_2^2 x_3^2 + x_2^2 x_1^2 + x_1^2 x_2^2 = \phi_2^4 - 2\phi_3\phi_2, \\
\rho y_3 &= x_3^3(x_2 + x_3) + x_2^3(x_2 + x_1) + x_1^3(x_2 + x_2) \\
&= \phi_3^2\phi_2 - 2\phi_2^3 - \phi_1\phi_3, \\
\rho y_4 &= x_2^2 x_3 x_3 + x_2^2 x_2 x_2 + x_2^2 x_2 x_2 = \phi_3\phi_2.
\end{align*}
\]

A simple elimination process of \( \psi_1, \psi_2, \psi_3 \) leads to the required relations between the \( \psi \)'s:

\[
(3) \quad \psi_1(\psi_2 + 2\psi_4) + 2\psi_2 - \psi_3^2 - \psi_4^2 + 4\psi_2\psi_4 - 2\psi_3\psi_4 = 0,
\]

which is a quadric cone \( Q \) with the vertex \( V(4, -2, -1, -1) \), as can easily be verified.

To a plane section of \( Q \) corresponds in \((x)\) a quartic which may be any of the reducible or irreducible types \((1)\). Of particular importance are, of course, the exceptional elements of the \((1, 6)\) correspondence between \( Q \) and \((x)\). In the first place for the intersections \( I(1, \omega, \omega^2), J(1, \omega^2, \omega) \) of \( \phi_1 = 0 \) and \( \phi_2 = 0 \) there is \( \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0 \), so that \( I \) and \( J \) are fundamental points in \((x)\). To the first neighborhoods of \( I \) and \( J \), \((1 + \alpha_1, \omega + \alpha_2, \omega^2 + \alpha_3) \) and \((1 + \alpha_1, \omega^2 + \alpha_2, \omega + \alpha_3) \), correspond on \( Q \) for \( \alpha_1 + \alpha_2 + \alpha_3 \neq 0 \),

\[
\begin{align*}
\rho y_1 &= 4(\alpha_1 + \alpha_2 + \alpha_3), \\
\rho y_2 &= -2(\alpha_1 + \alpha_2 + \alpha_3), \\
\rho y_3 &= -(\alpha_1 + \alpha_2 + \alpha_3), \\
\rho y_4 &= \alpha_1 + \alpha_2 + \alpha_3,
\end{align*}
\]

or the point \( V(4, -2, -1, 1) \). To a sextuple on \( \phi_1 = 0 \), distinct from \( I \) and \( J \), corresponds on \( Q \) the point \( T(2, 1, -2, 0) \). A plane \( p = \psi_1 + \lambda\psi_2 + \mu\psi_3 + \nu\psi_4 = 0 \) cuts \( Q \) in a conic \( K \) and the join \( \psi T \) in a point \( R \) on \( K \), to which corresponds in \((x)\) the quartic

\[
(4) \quad \phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (4 - 2\lambda - \mu + \nu)\phi_1\phi_3 + (2 + \lambda - 2\mu)\phi_3^2 = 0,
\]

which has \( \phi_1 = 0 \) as a double tangent. To a generic point \( R \) on \( \psi T \) correspond thus the first neighborhoods of \( I \) and \( J \) on \( \phi_1 = 0 \). To a plane
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\[ y_1 + \lambda y_2 + \mu y_3 + (2\lambda + \mu - 4)y_4 = 0 \]

through $V$, corresponds the quartic

\[ \phi_1^4 + (\mu - 4)\phi_2^2 \phi_3 + (2 + \lambda - 2\mu) \phi_2^2 = 0, \]

which clearly reduces to the product of two conics of the type $\phi_i^2 + k\phi_i = 0$. Thus a generic plane of the bundle through $V$ cuts $Q$ in two generatrices to which correspond in $(x)$ two conics of the symmetric pencil $\phi_i^2 + k\phi_i = 0$. To a tangent plane of $Q$ corresponds a double conic $(\phi_i^2 + k\phi_i)^2 = 0$. The tangent plane at $R$

\[ y_1 + 6y_2 + 4y_3 + 12y_4 = 0 \]

touches $Q$ along $VT$, and to this intersection of the tangent plane, $VT$ counted twice, corresponds in $(x)$ the quadruple line $\phi_1^4 = 0$.

3. Mapping of Intersections of the Quadric Cone. A generic surface $F_n$ cuts $Q$ in a space curve $C_{2n}$ to which corresponds in $(x)$ a symmetric curve $C'_n$ (curve in which the coordinates enter symmetrically). As $F_n$ cuts $VT$ in $n$ points, $\phi_1 = 0$ is, in general, an $M$-fold double tangent of $C'_n$. This also appears directly from the fact that in (1) $\phi_3^2$ appears in $y_1, y_2, y_3$, and $\phi_1$ is a factor of all other terms in which $\phi_3^2$ is not contained.

Conversely to a symmetric $n$-ic $C'_n$ in $(x)$ corresponds on $Q$ a curve, whose order can easily be determined in every case. For instance, when $C'_n$ does not pass through $I$ and $J$, which is the case when $C'_n$ contains the term $\phi_3^m, 3m = n$, then a generic quartic $C'_4$ cuts $C'_n$ in $12m$ points which form $2m$ sextuples. To these correspond on $Q, 2m$ points which lie on a plane of the corresponding conic $K$ of $C_n$, and which are the intersections of the curve $C$ on $Q$, corresponding to $C'_4$. The order of $C$ is therefore $2m$. The curve $C_{2m}$ on $Q$ is cut out by a surface $F_m$, which may possibly pass through generatrices of $Q$, or through the point $T$. For example when $C'_n$ is a sextic, then $F_m$ is a quadric which passes through a generatrix of $Q$, or is a quadric cone with its
vertex at $T$. This is in agreement with the counting of constants. The number $N+1$ of terms of a symmetric $n$-ic is equal to the number of positive integral solutions of the diophantine equation $\alpha + 2\beta + 3\gamma = n$ resulting from the general term $\phi^{a}\phi^{b}\phi^{c}$ of the $n$-ic, and is equal to the nearest positive integer contained in $(n+3)^{3}/12$. For $n = 6, N+1 = 7$.

The number of constants in $F_{1}$ is 10 which is reduced to 7 by the condition to pass through a generatrix of $Q$. To the intersection of $F_{2}$ with $Q$ corresponds in $(x)$ an octavic. But to the generatrix of $Q$, common to $F_{2}$, corresponds a symmetric conic as a factor of the octavic, so that a sextic remains as a residual curve.

More generally $F_{m}$ cuts $Q$ in a curve to which corresponds in $(x)$ a curve of order $4m$. In order that this reduce to $3m$ it is necessary that a factor of order $m$ split off. These factors are of the form $\phi^{a}\phi^{b}$, with $\alpha + \beta = m$, and $F_{m}$ must pass $\alpha$ times through $R$ and contains $\beta$ generatrices of $Q$. Thus in case of $n = 9, m = 3$, $F_{m}$ must be either a cubic cone with vertex at $T$, or a cubic surface through $T$ and a generatrix of $Q$. The condition to pass through $T$ and a generatrix of $Q$ absorbs 5 constants of $F_{3}$, and leaves 15 (14 effective) disposable constants. But through the intersection of a quadric and a cubic surface there are $\infty^{3}$ other cubics, so that there are $\infty^{12}$ linearly independent residual quintics on $Q$ to which corresponds in $(x)$ the same manifold of conics, which is in agreement with the number of solutions of the diophantine equation, in case of $n = 9$.

4. Symmetric Quartics. To a pencil of planes through a line $s$ cutting $Q$ in $A$ and $B$ corresponds in $(x)$ a pencil of quartics with the same double tangent $\phi_{i} = 0$ and with two sextuples $A'$ and $B'$, corresponding to $A$ and $B$, as base-points outside of the double tangencies at $I$ and $J$. When $s$ is tangent to $Q$, the quartics of the pencil all touch each other in the points of a sextuple. Now consider any two conics $K'$ and $K''$ on $Q$. The common tangent-planes of $K'$ and $K''$ envelope two cones. Through a generic
point of $Q$ there are two tangent-planes to each of these cones. To $K'$ and $K''$ correspond in $(x)$ two quartics $C_i'$ and $C_i''$. Every tangent plane of one of these cones cuts $Q$ in a conic which touches $K'$ and $K''$. To this conic corresponds a quartic which touches each $C_i'$ and $C_i''$ in points of a sextuple, one for each $C_i'$ and $C_i''$. Hence we may state the following theorem.

**Theorem.** Given two symmetric quartics $C_i'$ and $C_i''$, then there exist two $\infty^3$ systems of symmetric quartics of index 2, such that every quartic of each system has a sextuple contact (contact in points of a sextuple) with each $C_i'$ and $C_i''$.

To the intersection of a plane through $T$ with $Q$ corresponds in $(x)$ a symmetric cubic $\phi_0 + \lambda \phi_1 + \mu \phi_3 = 0$. Let $K'$ again be a conic not through $I$. Now $I$ is the vertex of a quadric cone through $K'$, whose tangent planes cut $Q$ in conics tangent to $K'$. To these correspond in $(x)$ cubics and a quartic respectively. This leads to the following theorem.

**Theorem.** For a symmetric quartic corresponding to a generic conic on $Q$ there exists a system of cubics of index 2 with the property of sextuple contacts with the quartic.*

5. **A Problem in Closure.** Let $K'$ and $K''$ be again two conics on $Q$ not intersecting in real points and $O$ a generic point in space. $O$ as a vertex determines with $K'$ and $K''$ two cones $C'$ and $C''$, which intersect $O$ in two other conics $L'$ and $L''$. Assume $O$ such that $C'$ and $C''$ have no real generatrix in common, moreover so that it is possible to construct a closed pyramid of $n$ faces inscribed to one cone, say $C'$, and circumscribed to $C''$. To the four conics $K', K'', L', L''$ correspond in $(x)$ four quartics $C_i', C_i'', D_i', D_i''$. To a conic cut out on $Q$ by a face of $P$ corresponds in $(x)$ a quartic which cuts each $C_i'$ and $D_i'$ in a sextuple and touches each $C_i''$ and $D_i''$ in points of a sextuple. As there

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* Such systems of sextuple tact cubics for the general quartic were established by A. Clebsch, Mathematische Annalen, vol. 3 (1871), pp. 45–75.
are \( \infty^1 \) such inscribed and circumscribed pyramids \( P \) the mapping upon \( (x) \) gives the following theorem.

**Theorem.** Let \( C'_i \) and \( C''_i \) be two fixed symmetric quartics in \( (x) \). Construct a quartic \( C^{(1)}_i \) with a sextuple contact with \( C'_i \), cutting \( C'_i \) in two sextuples \( S_1 \) and \( S_2 \). Through \( S_2 \) draw another quartic \( C^{(2)}_i \) with a sextuple contact with \( C''_i \), which cuts \( C'_i \) in another sextuple \( S_3 \). Through \( S_3 \) draw similarly a third quartic \( C^{(3)}_i \), cutting \( C'_i \) in a sextuple \( S_4 \), and so forth. Suppose that after drawing \( n \) such quartics, the last \( C^{(n)}_i \) through \( S_n \) cuts \( C'_i \) in a sextuple \( S_{n+1} \) which coincides with \( S_1 \). If this happens once then there exists an infinite number of such series of quartics with the closure property.

Moreover there exist two other fixed quartics \( D'_i \) and \( D''_i \) which are related to these series in precisely the same manner as \( C'_i \) and \( C''_i \).

**The University of Illinois**

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**CONGRUENCES OF LINES OF SPECIAL ORIENTATION RELATIVE TO A SURFACE OF REFERENCE**

**BY M. C. FOSTER**

1. **Introduction.** With each line \( l \) of a rectilinear congruence let us associate the point \( M \) in which \( l \) intersects a surface of reference \( S \). We refer \( S \) to any orthogonal system. Let \( \alpha, \beta, \gamma \) be the direction-cosines of \( l \) relative to the moving trihedral of \( S \) at \( M \), the \( x \)-axis being chosen tangent to the curve \( v = \text{const} \). By congruences of special orientation relative to \( S \), we shall mean those congruences for which the functions \( \alpha, \beta, \gamma \) are of a special form. The present paper is concerned primarily with the case when \( \alpha, \beta, \gamma \) are constant.

2. **Normal Congruences.** Relative to the moving trihedral the coordinates of any point \( P \) on \( l \) are

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