

## HAUSDORFF'S REVISED MENGENLEHRE

*Mengenlehre*. By Felix Hausdorff. Second edition. Berlin and Leipzig, Walter de Gruyter, 1927. 285 pp.

There are second editions and second editions. Some are merely second printings with misprints corrected and a few pages added to bring the subject matter up to date; others are complete revisions. Hausdorff's second edition is an extreme example of the latter type; here even the title has been revised the first edition having appeared in 1914 under the title *Grundzüge der Mengenlehre*.\*

Not only has the title of the book been abbreviated in this new edition, but the book itself has been reduced in size so that the number of pages is considerably less than two-thirds that of the first edition. This is due principally to a difference in the literary style of the two editions, the *Grundzüge* seeming extremely verbose when compared with the conciseness of the second edition. It is questionable whether the new edition will be as "teachable" as the old. Certainly it cannot be read with as much ease by a student approaching the subject for the first time, as so much more is left for him to supply for himself. Probably the best approach to the subject will be found to lie in a judicious selection of topics from both editions. We regret that we find in the second edition such a conscious effort on the part of the author to save space.

As in the first edition, there are two main topics considered, the first five chapters being concerned with the general theory of aggregates and the remaining chapters with the more special theory of point sets. In discussing the various chapters we shall give in general only the omissions from and the additions to the material contained in the first edition.

The *Vorbemerkungen* and Chapter 1 (pp. 9-24) contain the essential parts of the first two chapters of the *Grundzüge* in one-third the space required there. Among the topics omitted are symmetric sets and the principle of duality, and much less space is devoted to the "algebra" of sets. Among the topics retained the order is often changed; for example, a section on functions defined over a set is placed before the section on the fundamental ideas of the sum and common part† of two sets. Here the author has replaced the cumbersome  $\mathfrak{S}$ ,  $\mathfrak{D}$  notation of the *Grundzüge* by a more common notation: the sum  $S$  of two sets  $A$ ,  $B$  is indicated by  $S=A+B$ , the common part  $D$  by  $D=AB$ . As in the *Grundzüge*, the notation  $S=A+B$  is used if and only if  $D=0$ , thus differing from the usual notation which indicates the sum by  $S=A+B$  in all cases.

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\* Reviewed by Professor Henry Blumberg in this BULLETIN, vol. 72 (1920-21), pp. 116-129.

† *Durchschnitt*, translated by Blumberg as *section*. The most common expression is *product*, but Hausdorff uses this in a different sense.

Chapter 2 (pp. 25–41) on cardinal numbers and Chapter 3 (pp. 41–55) on order types correspond almost exactly to Chapters 3 and 4 respectively of the *Grundzüge*, although the number of pages has been reduced by half. This is partly accounted for by the omission of certain sections on ordered sets, the topic of ordered sets being given very brief consideration in this edition.

The fourth chapter (pp. 55–77) contains all the material on well-ordered sets and ordinal numbers that has been retained from Chapters 5 and 6 of the *Grundzüge*, where these topics covered five times the space. Practically all of Chapter 6 (partially-ordered sets, etc.) has been omitted, the only section that remains essentially as a unit being the section on general products and powers of ordered sets. The other topics considered are: the well-ordering theorem, the comparability of ordinal numbers, combination of ordinal numbers, and the Alephs.

In the fifth chapter (pp. 77–93) the author begins a comprehensive treatment of Borel and Suslin (or Souslin) sets, to which portions of Chapters 7 and 8 are also devoted. Suslin sets were first studied in 1917 by Michael Suslin (1894–1919), a Russian. These sets are a generalization of the well known Borel sets.

The remainder of the book (see however §40) is devoted to a discussion of sets of points considered as subsets of a metric space, that is, a space in which there exists a definition of the distance between each pair of points, subject to certain distance axioms. This is a departure from the viewpoint of the *Grundzüge*, where Hausdorff begins with a topological space, i.e., a space satisfying certain neighborhood axioms (*Umgebungsaxiome*). Having proved a number of theorems on the basis of this set of axioms, he then specializes the space step by step by adding additional restrictions, obtaining in turn a metric space, a euclidean space of  $n$  dimensions, and finally a euclidean plane, with appropriate groups of theorems in each case.

We do not quarrel with the author's decision to omit the discussion of euclidean spaces from the second edition, although we feel that the student loses something if he does not have brought to his attention the special methods applicable in euclidean spaces.

We do regret exceedingly, however, that he has treated only metric spaces when he might so easily have followed the outline of the *Grundzüge*. Instead of doing so, he proceeds as follows: having assumed that the space is metric, he defines (§22) a *neighborhood* of radius  $\rho$  of a point  $x$ , as the set of points whose distance from  $x$  is less than  $\rho$ . A system of neighborhoods defined thus will satisfy the *Umgebungsaxiome*. In proving many of the theorems that follow, the author makes use of this special system of neighborhoods, and in his proofs does not use all the properties of a metric space, but only those properties which are necessary to insure that the space contain a system of neighborhoods satisfying the *Umgebungsaxiome*. In other words, a number of theorems which are true in a topological space are proved by Hausdorff only for a metric space, when with a slight rearrangement of material he could have proved these theorems in all their generality. Since the author does not hesitate to assume whenever

necessary that his given metric space is complete, compact, or separable, if one of these hypotheses is needed to prove a particular theorem, it seems strange that he does not take the most general viewpoint throughout this part of the book, and assume merely that the point sets considered are subsets of a *topological* space, assuming it to be also metrical when and only when it is necessary. In this connection, compare §22 and §23 of the second edition with §2 and §3 of Chapter 7 of the first edition.

Chapter 6 (pp. 94–164) on point sets and Chapter 7 (pp. 164–193) on point sets and ordinal numbers correspond to Chapters 7 and 8 of the *Grundzüge*. Dyadic sets and sets of the first and second categories of Baire are given more prominence than in the first edition. A few pages are devoted to the idea of a locally connected (=connected *im kleinen*) set, an idea which Hahn and Mazurkiewicz were developing at the time of the appearance of the first edition. The results of the sixth chapter are obtained without any reference to ordinal numbers, while Chapter 7 is devoted to theorems in which this concept occurs or which are more easily proved by making use of this idea. Here the author shows that the totality of Suslin sets is not identical with the totality of Borel sets, but that there exist Suslin sets which are not Borel sets. In the concluding section, necessary and sufficient conditions are given in order that a Suslin set be a Borel set. In an appendix are given alternative proofs by Lusin of the theorems of this section, which proofs were communicated to the author by letter too late to be put in the proper place in the text. This is a striking proof of the current interest in this topic.

Chapter 8 (pp. 193–232) on correspondences between two spaces and Chapter 9 (pp. 232–275) on real functions comprise some material from Chapter 9 of the *Grundzüge*, but much of the material is new. The author first considers the subject of continuous curves: it is shown that the class of dyadic continua is identical with that of continuous curves; the Sierpinski and the Hahn-Mazurkiewicz characterizations of continuous curves are given. Next the subject of correspondences is taken up, under which are Lavrientieff's theorem on the extension of a continuous (1–1) correspondence, and a convenient table (p. 219) giving the image of various classes of point sets under correspondences of various types. Another section is devoted to the work of Hahn and R. L. Moore on the prime parts of a continuum. In the final section of Chapter 8 (§40), Hausdorff discusses briefly topological spaces, giving various sets of axioms used to distinguish the class of metric spaces from the more general class of topological spaces.

The ninth and concluding chapter gives a much more complete treatment of Baire's functions than was the case in the *Grundzüge*, and concludes with the theorems of Hahn and Sierpinski on the nature of the convergence set of a sequence of real continuous functions.

In addition to the omissions which we have mentioned above, the second edition omits the entire tenth chapter of the *Grundzüge* which treated content and measure of point sets and the applications of this theory to Lebesgue integrals.

The book closes with a list of the literature of the subject, a list of sources, and an index. Unlike the first edition, the list of sources gives

merely the original place of publication of the theorems cited; there are no supplementary remarks. The enlarged list of literature over that given in the first edition testifies to the growing interest in the subject of aggregate theory—an interest which the appearance of the *Grundzüge* did much to arouse and foster. The index is remarkably complete, and is an improvement over that of the first edition in that it contains also references to authors. Another improvement is the consecutive numbering of sections. The work of the publishers is excellent. We have discovered only a few unimportant misprints.

It is not within our province to criticize the author's choice of material. The book aims to be a textbook and not a report on the subject of aggregate theory, and therefore many topics have had to be omitted altogether, and in the case of those topics which have been included a choice of theorems has had to be made. Under such circumstances, the only criterion for the goodness of the author's choice is the taste of the individual reader.

The *Grundzüge* has been out of print for the past four years and we therefore heartily welcome the appearance of this new edition. We wish to state without qualification that this is an indispensable book for all those interested in the theory of aggregates and the allied branches of real variable theory and analysis situs. If we have seemed critical of some phases of the book, it is only because this excellent book is not better.

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#### GENERAL MATHEMATICS IN EUROPE

*Leçons de Mathématiques générales.* By L. Zoretti. With preface by P. Appell. Second edition. Paris, Gauthier-Villars, 1925. 16+788 pp.

*Nouveau Traité de Mathématiques générales.* By C. Fabry. Fourth edition. Vol. 2. Paris, Hermann, 1925. 276 pp.

*Einführung in die Elemente der höheren Mathematik.* By H. Hahn and H. Tietze. Leipzig, Hirzel, 1925. 12+331 pp.

*Cours Complet de Mathématiques Spéciales.* By J. Haag. Vol. 1, *Algèbre et Analyse.* 1914. 6+402 pp. Vol. 2, *Géométrie.* 1921. 7+662 pp. Vol. 3, *Mécanique.* 1922. 8+192 pp. Paris, Gauthier-Villars.

In the United States the term "general mathematics" has been used recently to denote a course for high school students. Such a course may have many advantages but it presents a troublesome problem to educational authorities when a pupil seeks entrance to college. It is difficult to determine whether certain definite entrance requirements have been met and it is sometimes equally puzzling to know whether the student is qualified to take certain courses in the college curriculum. General mathematics in a high school may consist of a mixture of arithmetic, algebra, geometry, and trigonometry in all sorts of proportions. The purpose usually is to prepare the pupils for practical living and college preparation is not the chief aim of the course.

On the continent of Europe the term "general mathematics" has been used to denote a course corresponding more nearly to those given in