NOTE ON INTERCHANGE OF ORDER OF LIMITS*

BY T. H. HILDEBRANDT

There exist a number of theorems giving necessary and sufficient conditions for the relation

\[ \lim_m \lim_n a_{mn} = \lim_n \lim_m a_{mn}, \]

\( \{a_{mn}\} \) being a double sequence of real numbers. Most of these theorems are not symmetric in \( m \) and \( n \), which is only natural, because as a rule wherever the interchange of order of limits is in question, there is information about the iterated limit in one order, and the existence of the limit in reverse order is desired. Nevertheless it may be of interest to deduce a condition which is symmetric in \( m \) and \( n \).

By way of notation, we shall assume that the symbol \( \lim_m \lim_n a_{mn} \) implies that for every \( m \), \( \lim_n a_{mn} \) exists. On the other hand, we shall use the symbol \( \lim_m \overline{\lim_n} a_{mn} \) with the implication that

\[ \lim_m \overline{\lim_n} a_{mn} = \lim_m \overline{\lim_n} a_{mn}. \]

Then, obviously, if we deduce necessary and sufficient conditions for the relation

\[ \lim_m \overline{\lim_n} a_{mn} = \lim_n \overline{\lim_m} a_{mn}, \]

those for the equality mentioned at the outset require in addition the assumption of the existence of \( \lim_n a_{mn} \) for every \( m \) and of \( \lim_m a_{mn} \) for every \( n \).

Let us deduce necessary conditions. Let \( \overline{\lim_m} a_{mn} = b_n \) and \( \lim_m a_{mn} = c_n \), and \( \lim_n b_n = \lim_n c_n = d \). Then for every \( \varepsilon > 0 \), there exists an \( n_\varepsilon \) such that when \( n \geq n_\varepsilon \), we have

\[ |b_n - d| \leq \varepsilon \text{ and } |c_n - d| \leq \varepsilon. \]

From the definition of \( b_n \) and \( c_n \) we have, for every \( n \), and

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every \( \epsilon > 0 \), an \( m_{en} \) (depending on \( \epsilon \) and \( n \)) such that when \( m \geq m_{en} \), we have

\[ c_n - \epsilon \leq a_{mn} \leq b_n + \epsilon. \]

By combining these two statements, we obtain the following. For every \( \epsilon > 0 \), there exists an \( n_{\epsilon} \) such that for every \( n_1 \geq n_{\epsilon} \), there exists an \( m_{en_1} \) such that for every \( m_2 \geq m_{en_1} \), we have

\[ d - 2\epsilon \leq a_{m_2n_1} \leq d + 2\epsilon, \]

i.e.,

\[ |a_{m_2n_1} - d| \leq 2\epsilon. \]

For future reference we shall call this Statement A. Similarly, from the fact that \( \lim_{m} \lim_{n} a_{mn} = d \), we have: For every \( \epsilon > 0 \) there exists an \( m_{\epsilon} \) such that for every \( m_1 \geq m_{\epsilon} \), there exists an \( n_{\epsilon m_1} \) such that for every \( n_2 \geq n_{\epsilon m_1} \), we have

\[ |a_{m_2n_2} - d| \leq 2\epsilon. \]

By combining these two statements into one, and replacing \( 4\epsilon \) by \( \epsilon \), we get the necessary condition desired, viz.: For every \( \epsilon > 0 \), there exists an \( n_{\epsilon} \) and an \( m_{\epsilon} \) such that for every \( n_1 \geq n_{\epsilon} \) and \( m_1 \geq m_{\epsilon} \), there exists an \( m_{en_1} \) and an \( n_{\epsilon m_1} \) such that for every \( m_2 \geq m_{en_1} \) and \( n_2 \geq n_{\epsilon m_1} \), we have

\[ |a_{m_2n_2} - a_{m_1n_1}| \leq \epsilon. \]

This condition is also sufficient. For suppose the condition satisfied. Then for a particular \( m_1 \) and \( n_2 \) chosen in accordance with the specifications, and for every \( n_1 \geq n_{\epsilon} \) and every \( m_2 \geq m_{en_1} \), we have

\[ a_{m_1n_2} - \epsilon \leq a_{m_1n_1} \leq a_{m_1n_2} + \epsilon. \]

From this we conclude that for every \( n_1 \geq n_{\epsilon} \), the greatest and the least of the limits \( b_n \) and \( c_n \) satisfy the conditions

\[ a_{m_1n_1} - \epsilon \leq c_{n_1} \leq b_{n_1} \leq a_{m_1n_2} + \epsilon. \]

This has as consequence that if \( n_1 \) and \( n_0 \) are greater than or equal to \( n_{\epsilon} \), then

\[ |b_{n_1} - b_{n_0}| \leq 2\epsilon, \quad |c_{n_1} - c_{n_1}| \leq 2\epsilon, \quad \text{and} \quad |c_{n_1} - b_{n_1}| \leq 2\epsilon. \]
Hence \( \lim_n c_n \) and \( \lim_n b_n \) exist and are equal, i.e. \( \lim_n \lim_m a_{mn} \) exists. From the symmetry of the condition, we conclude that \( \lim_m \lim_n a_{mn} \) exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, \( \lim_{mn} a_{mn} \), is the special case of our condition in which \( m_{en_1} \) and \( n_{en_1} \) are independent of \( n_1 \) and \( m_1 \) respectively, and can therefore be taken as \( m_e \) and \( n_e \), respectively.

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ON BOUNDED REGULAR FRONTIERS
IN THE PLANE*

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1. Introduction. The term regular frontier has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski|| has given necessary conditions for a continuum to be a regular frontier which is the frontier of every component of its complement.

* Presented to the Society, October 29, 1927.
† P. Urysohn, Mémoire sur les multiplicités Cantoriennes, Fundamenta Mathematicae, vol. 7, p. 98.
‡ A. Rosenthal, Teilung der Ebene durch Irreduzible Kontinua, Sitzungsberichte der Münchener Akademie, 1919.