Hence $\lim_n c_n$ and $\lim_n b_n$ exist and are equal, i.e. $\lim_n \lim_m a_{mn}$ exists. From the symmetry of the condition, we conclude that $\lim_m \lim_n a_{mn}$ exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, $\lim_{mn} a_{mn}$, is the special case of our condition in which $m_{en_1}$ and $n_{en_1}$ are independent of $n_1$ and $m_1$ respectively, and can therefore be taken as $m_*$ and $n_*$ respectively.

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ON BOUNDED REGULAR FRONTIERS
IN THE PLANE*

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1. Introduction. The term regular frontier has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski|| has given necessary conditions for a continuum to be a regular frontier which is the frontier of every component of its complement.

* Presented to the Society, October 29, 1927.
† P. Urysohn, Mémoire sur les multiplicités Cantoriennes, Fundamenta Mathematicae, vol. 7, p. 98.
‡ A. Rosenthal, Teilung der Ebene durch Irreducible Kontinua, Sitzungsberichte der Münchener Akademie, 1919.
In the present article Rosenthal's theorem is generalized to cover the case of two continua having a finite or enumerably infinite set of closed sets in common and irreducible between each pair. With this result and a theorem by the author† given elsewhere, it is possible to formulate necessary and sufficient conditions for a bounded continuum to be a regular frontier. The principal theorems are to be found in §§5–7.

2. Notation. Besides the ordinary notation of the aggregate theory the following special notation and terminology will be used.

The whole plane will be denoted by $Z$. If $A$ is a sub-set of $C$, the set of inner points of $A$ relative to $C$ will be denoted by $A^*$. If $A$ and $B$ are two sets in the plane $Z$ without common points and $C$ separates $A$ from $B$ (i.e., $C \cdot A = C \cdot B = 0$ and every continuum in the plane which contains points of both $A$ and $B$ also contains one or more points of $C$), we say that $C$ is an $S(A, B)$. If $F$ is a frontier set, a component of $Z - F$ which has the frontier $F$ will be called a principal component.

The statement "$C$ is a continuum irreducible between the sets $A$ and $B"$ will apply not only to the case that $A + B \subseteq C$, but also to the case that $C$ is irreducible between each point of $C \cdot A$ and each point of $C \cdot B$.

3. Some Auxiliary Theorems. Let $F = H + K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$, such that both $H$ and $K$ are irreducible between each pair. The following properties of $F$ are either well known or so easily established that the proofs are omitted.

(a) If the number of sets $\{\alpha\}$ is greater than two, both $H$ and $K$ are indecomposable.

(b) If $C$ is a sub-continuum of $H$ or of $K$, $F - C$ is connected.

(c) $H^* = F - K$ and $K^* = F - H$ are connected; also $H^* = H$ and $K^* = K$.

† W. A. Wilson, On irreducible cuts of the plane between two points. (To appear soon in Annals of Mathematics.)
If $F$ lies in a plane $Z$, certain properties expressing relations between $F$ and the rest of the plane can be proved without difficulty.

(d) If $H \cdot K$ has at least $n$ components, there are at least $n$ components of $Z - F$ which have frontier points on both $H^*$ and $K^*$. If $H \cdot K$ has an infinity of components, the components of $Z - F$ having frontier points on both $H^*$ and $K^*$ are enumerably infinite.

These statements are proved by adding to $H$ and $K$ those components of $Z - F$ which do not have frontier points on both $H^*$ and $K^*$ and applying a theorem of S. Straszewicz.t

(e) There is a bounded continuum $P$ which is an irreducible $S(H^*, K^*)$ and contains $H \cdot K$.

To establish this we observe that, since $H^*$ and $K^*$ are connected, there is a bounded continuum $Q$ which is an $S(H^*, K^*)$ by a theorem of Knaster and Kuratowski.‡ But by a theorem of Mazurkiewicz,§ $Q$ contains a sub-continuum $P$ which is an irreducible $S(H^*, K^*)$. Moreover $P \supset H \cdot K$, since every point of $H \cdot K$ is a common limiting point of $H^*$ and $K^*$.

It may be added that every bounded irreducible $S(H^*, K^*)$ is a continuum containing $H \cdot K$.

4. LEMMA. Let $F = H + K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$ between each pair of which both $H$ and $K$ are irreducible. Let $R$ be a component of $Z - F$ such that $\overline{R}$ contains a bounded continuum containing points of two or more sets $\alpha$ but no other points of $F$. Then $F$ is the frontier of $R$.

PROOF. The principles of inversion permit us to demonstrate the lemma on the assumption that $R$ is bounded. Let

C be the continuum referred to above. Then $R$ contains a closed polygon $J$ (whose interior is $I$) so large that $C - C \cdot I$ has no component containing points of more than one $\alpha$. This is easily seen since the number of sets $\alpha$ is finite. On the other hand, if $\alpha_1$ and $\alpha_2$ are two sets met by $C$, there are sub-continua $A_1$ and $A_2$ of $C - C \cdot I$ irreducible between $\alpha_1$ and $\alpha_2$, respectively, and $J$, and $A_1 \cdot A_2 = 0$. Let $M$ be an arc of $J$ irreducible between $A_1$ and $A_2$. Then $Q = A_1 + M + A_2$ is irreducible between $\alpha_1$ and $\alpha_2$ and does not meet $F - (\alpha_1 + \alpha_2)$.

Since $Q$ and $H$ are both irreducible between $\alpha_1$ and $\alpha_2$ and have only points of these sets in common, while $Q$ is not the union of two indecomposable continua, it follows by an extension of Rosenthal's theorem† that $Z - (Q + H)$ has two principal components $S_1$ and $S_2$. Similarly, let $T_1$ and $T_2$ be the principal components of $Z - (Q + K)$.

As $K^*$ is connected, it lies in but one component of $Z - (Q + H)$; suppose that $K^* \cdot S_1 = 0$. Since $(F + Q) \cdot S_1 = 0$ and $S_1$ has frontier points on $M$, $R \cdot S_1 \neq 0$ and $S_1 \subseteq R$. Likewise either $T_1$ or $T_2$, say $T_1$, is a part of $R$. As $F = H + K$, every point of $F$ is a frontier point of either $S_1$ or $T_1$, and a fortiori of $R$.

5. Theorem. Let $F = H + K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite number $n$ of closed sets $\{\alpha\}$ between each pair of which both $H$ and $K$ are irreducible. Then the number of principal components of $Z - F$ is at least $n$.

Proof. Let $R_i$ be any component of $Z - F$ having frontier points on both $H^*$ and $K^*$, let $P$ be a bounded irreducible $S(H^*, K^*)$, and let $P_i = P \cdot R_i$. Suppose that $P_i$ has a sub-continuum containing points of more than one set $\alpha$ for only $k$ values of $i$, say $i = 1, 2, \ldots, k$, where $k < n$. We first show that $k \geq 1$. For, if $\alpha'$ denotes any $\alpha$ and $\alpha''$ the sum of the remaining sets $\alpha$, the continuum $P$ contains a connected

set having no points of \(\alpha'\) or \(\alpha''\), but limiting points on both. This connected set must be a part of some \(P_i\).

In some work done elsewhere\(^1\) it was shown that there is a continuum \(Q\), which is a bounded irreducible \(S(H^*, K^*)\), constructed as follows: \(Q\) is the union of \(k\) arcs \(\{A_i\}\), each \(A_i\) lying in \(R_i\), \(i = 1, 2, \ldots, k\), and \(k\) continua \(\{B_i\}\), such that \(B_i \cdot B_{i'} = 0\) if \(i \neq i'\), each \(B_i\) is irreducible between an end of \(A_i\) and one of \(A_{i+1}\) \((i + 1 = 1, \text{if } i = k)\) and meets no other points of \(\sum_i A_i\) and \(H \cdot K \subset \sum_i B_i \subset P\). Moreover, in the demonstration referred to, the circle used may be replaced by a polygon so large that, if \(I_i\) is its interior, no component of \(P_i - P_i \cdot I_i\) has points on two sets \(\alpha\). Consequently any connected sub-set of any \(B_i\) having limiting points on two sets \(\alpha\), but no points on \(F\), must lie in some \(R_i\), \(i > k\).

Since \(n > k\), some \(B_i\) contains points of two or more sets \(\alpha\). Hence for some \(i > k\), there is a connected sub-set \(j\) of \(B_i\) having limiting points on two sets \(\alpha\), but containing no points of \(F\).\(^2\) Then \(j\) lies in some \(P_i\), where \(i > k\) by the last part of the previous paragraph. Therefore \(P_i\) contains points of more than one set \(\alpha\), contrary to the assumption at the beginning of the proof.

Therefore, for at least \(n\) values of \(i\), \(P_i\), and consequently \(R_i\), contains a continuum having points on more than one set \(\alpha\). Then \(\S 4\) shows that \(F\) is the frontier of each such \(R_i\). Hence the theorem is proved.

**Corollary.** Let \(F = H + K\) be the union of two bounded continua and for every integer \(n\) let \(H \cdot K\) be the sum of \(n\) closed sets such that both \(H\) and \(K\) are irreducible between each pair. Then the number of principal components of \(Z - F\) is infinite.

\(^1\) See reference to paper by the author under \(\S 1\), §§6 and 7. The hypothesis in these sections requires \(R\) to be a principal component of \(Z - F\), but the demonstration only requires that \(R\) have frontier points on both \(H^*\) and \(K^*\).

\(^2\) Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24, Theorem 1.
6. **Theorem.** For the bounded decomposable continuum $F$ to be the frontier of exactly $n$ components of its complement, it is necessary and sufficient that $F$ be the union of two continua $H$ and $K$ such that $H \cdot K$ is the sum of $n$, but of no finite number greater than $n$, closed sets between each pair of which both $H$ and $K$ are irreducible.

**Proof.** If $F$ is decomposable and $Z-F$ has $n$ principal components, it has been shown elsewhere† that $F$ is the union of two continua $H$ and $K$ such that $H \cdot K$ is the sum of a finite number, greater than or equal to $n$, of closed sets between each pair of which both $H$ and $K$ are irreducible. On the other hand, if $H \cdot K$ can be expressed as the sum of $n$ closed sets between each pair of which both $H$ and $K$ are irreducible, there are at least $n$ principal components of $Z-F$ by §5.

The combination of these statements gives the theorem.

**Remarks.** The decomposition of $H \cdot K$ into $n$ closed sets given above is unique. For, if there were two different decompositions into $n$ closed sets with the properties assigned, say $H \cdot K = \sum_i^n \alpha_i$ and $H \cdot K = \sum_i^n \beta_i$, some $\beta_i$, say $\beta_1$, would contain points of more than one set $\alpha_i$. Let $\beta_1 \cdot \alpha_1 \neq 0$ and let $\beta_{11} = \beta_1 \cdot \alpha_1$ and $\beta_{12} = \beta_1 \cdot (H \cdot K - \alpha_1)$. Then $H \cdot K = \beta_{11} + \beta_{12} + \sum^n \beta_i$ is a decomposition into $n+1$ closed sets between each pair of which both $H$ and $K$ are irreducible. We then have the contradiction that $Z-F$ has at least $n+1$ principal components.

7. **Theorem.** For the bounded decomposable continuum $F$ to be the frontier of an infinity of components of its complement, it is necessary and sufficient that $F$ be the union of two continua $H$ and $K$ such that for every integer $n$ the set $H \cdot K$ is the sum of $n$ closed sets between each pair of which $H$ and $K$ are irreducible.

**Proof.** The necessity of these conditions was shown in

† See reference to paper by the author under §1.
the proof of the theorem referred to in §6. That they are sufficient follows from §5, Corollary.

8. Two Examples. It might be inferred that the statements of the theorems in §§6 and 7 are unnecessarily complicated and that, if $H \cdot K$ is the sum of an infinite set of closed sets between each pair of which both $H$ and $K$ are irreducible, then $Z-F$ has an infinite set of principal components. This is not in general true. The following examples show the existence of two continua $H$ and $K$ having these properties, but such that $Z-F$ has no principal component in Ex. I and but two principal components in Ex. II.

Example I. Let $Q = OABC$ be a closed rectangle, such that the lengths of $OA$ and $AB$ are 1 and $1/2$ unit, respectively. Let $q$ be the frontier of $Q$. Let $M$ be a Cantor set extending from $O$ to $A$, whose complementary open intervals $\{I_n\}$ are ordered according to size.

Now let a finite ordered set of closed rectangles, each of width $d$ and of length greater than $d$, such that each rectangle has in common with the one preceding and the one following two different squares of side $d$ but no point in common with any other rectangle, be called a band of width $d$. It is

\[ \text{Figure 1} \]

easily seen that there is in \( Q \) a unique band \( B_1 \) of width \( 1/3 \), which is contiguous to all points of \( q-I_1 \) and whose frontier \( q_1 \) is the union of \( q-I_1 \) and a broken line \( b_1 \) meeting \( q \) only in the end-points of \( I_1 \). Then \( Q-B_1 \) consists of \( I_1 \) and a simply connected region \( E_1 \) whose frontier is \( I_1+b_1 \); let \( G_1=Q-B_1 \).

Likewise, in \( B_1 \) there is a unique band \( B_2 \) of width \( 1/3^2 \) whose frontier \( q_2 \) is the union of \( q_1-I_1 \) and a broken line \( b_2 \) meeting \( q_1 \) only in the end-points of \( I_2 \). Then \( G_2=B_1-B_2 \) consists of \( I_2 \) and a region \( E_2 \) whose frontier is \( I_2+b_2 \). This construction can be repeated indefinitely; in the figure the shaded area is \( G_1+G_2+G_3 \); the unshaded with its border is \( B_2 \).

Set \( H=Q-\bigcap_{n=1}^{\infty} G_n=\bigcup_{n=1}^{\infty} B_n \). Obviously \( H \) is a continuum.

The Cantor set \( M \) may be regarded as the sum of an infinity of closed sets without common points. These are the set \( O+A \), an enumerable set of sets each consisting of the end-points of an interval \( I_n \), and a non-enumerable set of sets each of which is one of the other points of \( M \). Let \( \alpha \) be any of these sets. It is comparatively easy to show that \( H \) is irreducible between each pair of sets \( \alpha \).

Let \( K \) be the continuum symmetrical to \( H \) with respect to \( OA \). Then \( K \) is also irreducible between each pair of sets \( \alpha \) and \( H \cdot K=\Sigma \alpha \). But each component of \( Z-F \) has for its frontier a pair of continua of condensation of \( H \) and \( K \).

**Example II.** This is a variation of Ex. I. Take a closed rectangle \( Q \) whose length and width are \( 43/27 \) units and 1 unit respectively. Let \( ab \) be one side and let the points \( c, d, e \) and \( f \) divide \( ab \) into five intervals of lengths \( ac=1/3; cd=2/27; de=1; ef=2/27; fb=1/9 \). Let \( \{ I_k \} \) be an enumerable set of open intervals divided into three classes as follows: \( \{ I_{sn} \} \) is the set of complementary intervals of a Cantor set \( M \) in the interval \( de \) ordered as in Ex. I; \( \{ I_{sn-2} \} \) and \( \{ I_{sn-1} \} \) are two other sets, of which the first are \( I_1=cd \) and \( I_2=ef \), and the others will be defined later. Let \( q \) be the frontier of \( Q \).

For \( k=1, 2, 3 \), let \( B_k, q_k, b_k, E_k, \) and \( G_k \) be defined as in Ex. I. For \( n=2 \), let \( I_k=I_{3n-2}=I_k \) be an open interval of length \( 1/9 \) at the center of one of the longest segments of
$b_1$ perpendicular to, but not meeting, $ab$. Then $B_4$ is a band of width $1/3^4$ contiguous to $q_3 - I_4$. This gives $q_4$, $b_4$, $E_4$, and $G_4$. In this case, we note that $E_1 + G_4$ is a simply connected region whose frontier is $I_1 + (b_1 - I_4) + b_4$. In like manner, we let $I_5$ be an open interval of length $1/9$ at the center of one of the longest segments of $b_2$ perpendicular to and not meeting $ab$ and define $B_5$, $q_5$, $b_5$, $E_5$, and $G_5$. Here $E_2 + G_5$ is a simply connected region whose frontier is $I_2 + (b_2 - I_5) + b_5$.

We now return to $I_6$, which has been defined above. Let this process be repeated indefinitely. Each $I_{3n}$ lies on $de$; each $I_{3n-2}$ is an open interval of length $1/9$ at the center of one of the longest segments of $b_{3n-2}$ perpendicular to and not meeting $ab$; and similarly for $I_{3n-1}$. In the figure the shaded

![Figure 2](image_url)

areas are $E_1 + G_4$ and $E_5$; the narrower white area is $B_4$; and the other white area is $E_2$. For every value of $k$, $Q$ is the union of $B_k$ and a finite number of the sets $G_k$. No two of the sets $G_{3n}$ or of the corresponding regions $E_{3n}$ have common points. For each $n$, $E_1 + G_4 + \cdots + G_{3n-2}$ and $E_2 + G_5 + \cdots + G_{3n-1}$ are simply connected regions whose frontiers
are the union of $I_1$ and $I_2$, respectively, with a broken line interior to $Q$.

Let $H = Q - \sum_{k=1}^{\infty} G_k = \prod_{k=1}^{\infty} B_k$. Obviously $H$ is a continuum. The set $ac+fb+M$ is a closed set which may be regarded as the sum of an infinity of closed sets without common points. These are the set $ac+fb$, an enumerable set of sets each consisting of the end-points of an interval $I_{3n}$, and a non-enumerable set of sets each of which is one of the other points of $M$. Let $\alpha$ denote any of these sets. It is comparatively easy to show that $H$ is irreducible between each pair of sets $\alpha$.

Reflect the above figure about $ab$ and denote corresponding sets by primes. Then $K = H'$ is a continuum irreducible between each pair of sets $\alpha$ and $H \cdot K = \sum \alpha$. The components of $Z - F$, where $F = H + K$, are the exterior of the rectangle $Q + Q'$, the enumerable set of regions $G_{3n} + G_{3n}'$, each of which has as its frontier a pair of continua of condensation of $F$, and two other regions. These are $\sum_{n=1}^{\infty} G_{3n-2} + \sum_{n=1}^{\infty} G_{3n-2}'$ and $\sum_{n=1}^{\infty} G_{3n-1} + \sum_{n=1}^{\infty} G_{3n-1}'$. Each of these has the frontier $F$ and is therefore a principal component of $Z - F$.

The essential difference between the two examples is that in Ex. I, $H \cdot K$ cannot be divided into any finite number of closed sets between each pair of which $H$ and $K$ are irreducible, while in Ex. II, $H \cdot K$ can be divided into precisely two, but no more, such sets, namely the sets $ac+fb$ and $M$. Thus Ex. II is in strict accordance with §§6 and 7.

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