THE POLAR CURVES OF PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS*

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By imitating the proofs in Fine's *College Algebra* (pp. 460–462) and Veblen and Young's *Projective Geometry* (vol. I, pp. 255–256) we can readily show that also in the Galois fields of order \( p^n \) (\( p \) a prime integer) we have Taylor's expansion

\[
f(x + \lambda X, y + \lambda Y, z + \lambda Z) = f(x, y, z) + \frac{\lambda}{1!} (f'_x X + f'_y Y + f'_z Z) + \frac{\lambda^2}{2!} (f''_x X + f''_y Y + f''_z Z)^{(2)} + \cdots + \frac{\lambda^r}{r!} (f'^r_x X + f'^r_y Y + f'^r_z Z)^{(r)} + \cdots + f(X, Y, Z) = 0,
\]

where \( (f'_x X + f'_y Y + f'_z Z)^{(i)} \) is symbolic for an expression containing derivatives of the \( i \)th order, and \( f(x, y, z) = 0 \) is an algebraic curve of order \( n \). In the above expansion we must take all the derivatives as though \( p \) were not a modulus, cancel out common factors from numerators and denominators, and then set \( p = 0 \).

The \( r \)th polar of \((X, Y, Z)\) with respect to \( f(x, y, z) = 0 \) is

\[
\frac{1}{r!} (f'^r_x X + f'^r_y Y + f'^r_z Z)^{(r)} = 0.
\]

In particular the \( r \)th polar of \((1, 0, 0)\) is \((1/r!) \partial^r f(x, y, z) / \partial x^r = 0 \). We suppose first of all that \( n \) has the value

\[
n = \alpha p^m + \beta p^{m-1} + \cdots + \gamma p^2 + \delta p + \epsilon,
\]

\( \epsilon \neq 0, \quad p = \epsilon + \zeta, \quad \zeta \neq 0 \).

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We can write the polars of \((1, 0, 0)\) by a sort of detached coefficients, underlining the coefficients that have \(p\) as a factor, as follows:

\[
(1/1!) [n, n-1, n-2, \ldots, n-\varepsilon, n-\varepsilon-1, \ldots, n-p-\varepsilon, n-p^2-\varepsilon, \ldots, 3, 2, 1] = 0,
\]

\[
(1/2!) [n(n-1), (n-1)(n-2), \ldots, (n-\varepsilon+1)(n-\varepsilon), (n-\varepsilon)(n-\varepsilon-1), \ldots, (n-p-\varepsilon+1)(n-p-\varepsilon), (n-p-\varepsilon)(n-p-\varepsilon-1), \ldots, (n-p^2-\varepsilon-1)] = 0,
\]

\[
(1/(\varepsilon+1)!) [n(n-1)(n-2) \ldots (n-\varepsilon), (n-1)(n-2) \ldots (n-\varepsilon)(n-\varepsilon-1) \ldots (n-2\varepsilon), (n-\varepsilon-1)(n-\varepsilon-2) \ldots (n-2\varepsilon)(n-2\varepsilon-1), \ldots, (\varepsilon+1)!] = 0,
\]

\[
(1/p!) [n(n-1) \ldots (n-\varepsilon) \ldots (n-p+1) \ldots, p!] = 0,
\]

where \((n-\lambda) \ldots (n-\lambda-i)\) stands for all the terms of the same \((n-\lambda-i-1)\) power, which then have this common factor in their coefficients. From the above polars we see that the \(\varepsilon\)th polar has at \((1, 0, 0)\) a tangent having \((\varepsilon+1)\)-point contact if \((1, 0, 0)\) is not on \(f(x, y, z) = 0\), otherwise a multiple point of order \(\varepsilon+1\). The \((\varepsilon+1)\)th polar, \((\varepsilon+2)\)th, \ldots, \((p-1)\)th polar all have multiple points of order \(\varepsilon+1\) at \((1, 0, 0)\). Similarly the \((p+\varepsilon+1)\)th polar, \((p+\varepsilon+2)\)th, \ldots, \((2p-1)\)th have at \((1, 0, 0)\) multiple points of order \(\varepsilon+1\); also the \((2p+\varepsilon+1)\)th polar points of order \((3p-1)\) \(h, \ldots, \), the \((\theta p^1 + \ldots + \phi p + \varepsilon + 1)\)th polar points of order \((\theta p^1 + \ldots + \phi p + p - 1)\)th, etc. Moreover we note that if any one of the polar curves that have multiple points at \((1, 0, 0)\) is a curve of degree \(\varepsilon+1\), then this polar curve is degenerate. Thus for \(p=2, n=2^2+1, \varepsilon=1\), we find the 2d
polar is degenerate; for $p = 3$, $n = 3 + 1, \epsilon = 1$, we find again the 2d polar is degenerate.

If $n = \alpha p^m + \beta p^{m-1} + \cdots + \gamma p^2 + \delta p$, i.e. $\epsilon = 0$ in $n$, then all the polars of $(1, 0, 0)$ pass through $(1, 0, 0)$ whether or not this point lies on $f(x, y, z) = 0$.

If $n < p$ we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

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1. Introduction. Consider any square matrix $A$, real or complex, of order $n$. If $I$ is the unit matrix, $A - \lambda I$ is called the characteristic matrix of $A$; the determinant of the characteristic matrix is called the characteristic determinant of $A$; the equation obtained by equating this determinant to zero is called the characteristic equation of $A$; and the roots of this equation are called the characteristic roots of $A$. If $A$ happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if $A$ is Hermitian its characteristic roots are all real; if $A$ is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if $A$ is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if $A$ is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if $\alpha + i\beta$ is a characteristic root of a real matrix $A$, and if $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ are the characteristic roots (all real) of the symmetric matrix $\frac{1}{2}(A + A')$, then $\rho_1 \geq \alpha \geq \rho_n$. The extension to the case where the elements of $A$ are com-

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