polar is degenerate; for \( p = 3 \), \( n = 3 + 1 \), \( \epsilon = 1 \), we find again the 2d polar is degenerate.

If \( n = \alpha p^m + \beta p^{m-1} + \cdots + \gamma p^a + \delta p \), i.e. \( \epsilon = 0 \) in \( n \), then all the polars of \((1, 0, 0)\) pass through \((1, 0, 0)\) whether or not this point lies on \( f(x, y, z) = 0 \).

If \( n < p \) we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

BY E. T. BROWNE

1. Introduction. Consider any square matrix \( A \), real or complex, of order \( n \). If \( I \) is the unit matrix, \( A - \lambda I \) is called the characteristic matrix of \( A \); the determinant of the characteristic matrix is called the characteristic determinant of \( A \); the equation obtained by equating this determinant to zero is called the characteristic equation of \( A \); and the roots of this equation are called the characteristic roots of \( A \). If \( A \) happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if \( A \) is Hermitian its characteristic roots are all real; if \( A \) is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if \( A \) is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if \( A \) is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if \( \alpha + i\beta \) is a characteristic root of a real matrix \( A \), and if \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n \) are the characteristic roots (all real) of the symmetric matrix \( \frac{1}{2}(A + A') \), then \( \rho_1 \geq \alpha \geq \rho_n \). The extension to the case where the elements of \( A \) are com-

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plex was made by Hirsch* in 1902. In 1904 Bromwich† further extended the theorem as follows: If \( \alpha + i\beta \) is a characteristic root of a matrix \( A \) whose elements are real or complex, and if \( \rho_1, \rho_2, \cdots, \rho_n \) are the characteristic roots (all real) of \( \frac{1}{2}(A + A') \) and \( i\mu_1, \cdots, i\mu_n \) are the characteristic roots of \( \frac{1}{2}(A - A') \), then \( \alpha \) lies between the greatest and the least of \( \rho_1, \cdots, \rho_n \), and \( |\beta| \) does not exceed the greatest of \( |\mu_1|, \cdots, |\mu_n| \).

In some cases the theorems just cited give very good limits for the characteristic roots of a matrix, while in other cases the limits are not so restricted. Thus in the case of a real orthogonal matrix these theorems may merely state that the characteristic roots lie in the square \( x = \pm 1, y = \pm 1 \). In this paper we shall give a criterion which in some cases, notably in the case of a real orthogonal matrix, give more restricted limits than the theorems above.

2. Reduction of a Matrix to a Semi-Unitary Form. Let \( A \) be any square matrix of order \( n \). Then \( AA' \) is Hermitian and there exists a unitary matrix \( K \) (that is, \( KK' = I \)) such that

\[
KA K' = M,
\]

where \( M \) is zero except in the diagonal, and the elements in the diagonal are the (real) characteristic roots \( \rho_1, \rho_2, \cdots, \rho_n \) of \( AA' \). We may write

\[
(1) \quad M = \kappa \bar{A} \kappa' \kappa A' \kappa' = B \bar{B}',
\]

where

\[
(2) \quad B = \kappa A \kappa'.
\]

From (1) the elements \( b_{ij} \) of \( B \) evidently satisfy the conditions

\[
(3) \quad \sum_{t} b_{it} \bar{b}_{jt} = \rho_{ij}, \quad (i, j = 1, \cdots, n),
\]

where $\delta_{ij}$ is the Kronecker symbol, and equals 1 if $i=j$; 0 if $i \neq j$. In view of the conditions (3) we shall say that $B$ is in a semi-unitary (semi-orthogonal, if $B$ is real) form.

If $\rho_i = 1$, $(i=1, \cdots, n)$, $B$ is unitary. We may then state the following theorem.

**Theorem I.** If $A$ is any square matrix of order $n$ there exists a unitary matrix $\kappa$ such that $\kappa A \kappa' = B$, where $B$ is in a semi-unitary form.

If $M$ is of rank $r$, $\kappa$ may be so chosen that $\rho_i > 0$, $(i=1, \cdots, r)$; $\rho_i = 0$, $(i=r+1, \cdots, n)$. Since $\rho_i = \sum_t b_{it} \delta_{tt} = 0$, $(i=r+1, \cdots, n)$, evidently $b_{tt} = 0$, $(i=r+1, \cdots, n; \ t=1, \cdots, n)$; that is, the last $n-r$ rows of $B$ consist entirely of zeros, so that $B$ is of rank at most $r$. Hence, $B$ must be of rank exactly $r$. Since the rank of $A$ equals the rank of $B$, and the rank of $AA'$ equals the rank of $M$, incidentally we have given a proof of the following well known theorem.

**Theorem.** If $A$ is any square matrix of order $n$, the ranks of $A$ and $AA'$ are the same.*

3. *The Characteristic Roots of $AA'$.** Referring to the matrix $B$ defined as in (1) and (2), let us form a non-singular matrix $C = (c_{ij})$ by replacing the zeros in the last $n-r$ rows of $B$ by elements $(x_{s1}, x_{s2}, \cdots, x_{sn}) \neq (0, 0, \cdots, 0)$, such that

\[
\sum_{t}^{1, \cdots, n} b_{st} \bar{x}_{st} = 0, \quad (i = 1, \cdots, r; \ s = 1, \cdots, n - r),
\]

and, moreover, such that

\[
\sum_{t}^{1, \cdots, n} x_{it} \bar{x}_{jt} = 0, \quad (i, j = 1, \cdots, n - r; \ i \neq j).
\]

Thus, we may find $(\bar{x}_{11}, \bar{x}_{12}, \cdots, \bar{x}_{1n})$ by determining a non-zero solution of the $n-r$ linear homogeneous equations (4).

Having obtained $(x_{11}, \cdots, x_{1n})$ we may proceed to find

(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) by adjoining to the system (4) the additional linear homogeneous equation

$$\sum_{t}^{1, \cdots, n} x_{1t}x_{2t} = 0;$$

and so on. If \( \sum_{t}^{1, \cdots, n} c_{ii} \bar{x}_{tt} = \rho_i, (i = 1, \cdots, n) \), then \( \rho_i > 0 \) and if we write

$$\chi_{ij} = \frac{c_{ij}}{(\rho_i)^{1/2}}, \quad (i, j = 1, \cdots, n),$$

the matrix \( \chi \) thus obtained is a unitary matrix. It is evident from the manner in which \( \chi \) was built up that \( B\bar{\chi}' \) is zero except in the diagonal. The elements in the last \( n - r \) places in the diagonal are also zero, while those in the first places are \( (\rho_i)^{1/2} \), the square roots of the characteristic roots of \( A\bar{A}' \).

Since \( B\bar{\chi}' \) is real and symmetric, the characteristic roots of

$$N = \chi\bar{B}'B\bar{\chi}' = (B\bar{\chi}')^2$$

are the squares of the characteristic roots of \( B\bar{\chi}' \), and are therefore the characteristic roots of \( A\bar{A}' \). But

$$N = \chi\bar{B}'B\bar{\chi}' = \chi\bar{\kappa}\bar{A}'\bar{\kappa}'\chi \bar{\chi}' = \chi\kappa\bar{A}'\bar{\kappa}'\bar{\chi} = \psi\bar{A}'A\psi',$$

where \( \psi \) is the unitary matrix \( \chi\kappa \). Thus it follows* that the characteristic roots of \( \bar{A}'A \) are the same as those of \( N \) and therefore of \( A\bar{A}' \). Hence we have the following theorem.

**THEOREM II.** If \( A \) is any square matrix of order \( n \) the characteristic roots of \( A\bar{A}' \) are the same as the characteristic roots of \( \bar{A}'A \).

Since the unitary matrices \( \kappa, \chi \) above are such that

$$\kappa A\kappa' = B, \text{ and } B\bar{\chi}' = \chi\bar{B}' ,$$

it follows at once that

$$\kappa A\kappa'\bar{\chi}' = B\bar{\chi}' = \chi\bar{B}' = \chi\kappa\bar{A}'\kappa'.$$

Hence

\[ \bar{K}'X'K = A'. \]

Writing \( \bar{K}'X'K = \phi \), we have the following theorem.

**Theorem III.** If \( A \) is any square matrix of order \( n \) there exists a unitary matrix \( \phi \) such that

\[ \phi A \phi = A'. \]

In this connection compare Hilton, *Homogeneous Linear Substitutions*, Ex. 6, p. 124.

Since from (5)

\[ A \phi = \bar{\phi}'A' = (A \bar{\phi})', \]

\( A \phi \) is Hermitian, so that we have the following theorem.

**Theorem IV.** If \( A \) is any square matrix of order \( n \), there exists a unitary matrix \( \phi \) such that \( A \phi \) is Hermitian.

4. *The Characteristic Roots of \( A \).* From (2) the characteristic roots of \( A \) are evidently the same as the characteristic roots of \( B \). Suppose then that \( \lambda \) is a characteristic root of \( B \) so that there exists a set \( (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0) \) such that

\[ \sum_{i}^{1, \ldots, n} b_{ii}x_i = \lambda x_i, \quad (i = 1, \ldots, n). \]

Taking the conjugates of both members of each of these equations, we have

\[ \sum_{i}^{1, \ldots, n} \bar{b}_{ii} \bar{x}_i = \bar{\lambda} \bar{x}_i, \quad (i = 1, \ldots, n). \]

Multiplying corresponding equations in (6) and (7), member for member, and summing as to \( i \), we find

\[ \sum_{i,t}^{1, \ldots, n} \left[ \sum_{i}^{1, \ldots, n} b_{ii} \bar{b}_{tt} \right] x_{it} \bar{x}_i = \lambda \bar{\lambda} \sum_{i}^{1, \ldots, n} x_{it} \bar{x}_i; \]

that is

\[ \sum_{i}^{1, \ldots, n} \rho_{it} x_{it} \bar{x}_i = \lambda \bar{\lambda} \sum_{i}^{1, \ldots, n} x_{it} \bar{x}_i. \]
Let $G$ be the largest and $s$ the smallest of the characteristic roots of $AA'$. Then

$$
\lambda \bar{\lambda} \sum x_i \bar{x}_i \leq G \sum x_i \bar{x}_i,
$$

so that $\lambda \bar{\lambda} \leq G$. Similarly, $\lambda \bar{\lambda} \geq s$; i.e.,

$$
s \leq \lambda \bar{\lambda} \leq G.
$$

In particular, if $A$ is unitary so that $A \bar{A}' = I$, then $G = s = 1$, so that $1 \leq \lambda \bar{\lambda} \leq 1$; i.e., $\lambda \bar{\lambda} = 1$, as is well known. Hence we have the following theorem.

**Theorem V.** If $\lambda$ is a characteristic root of a square matrix $A$ and if $G$ and $s$ are respectively the largest and the smallest characteristic roots of $A \bar{A}'$, then

$$
s \leq \lambda \bar{\lambda} \leq G.
$$

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