The proof is immediate, for by the Hamilton-Cayley theorem

\[ \delta(R(x)) = 0, \quad \delta'(S(x)) = 0. \]

Since \( \mathfrak{A} \) is isomorphic with the algebra of matrices \( R(x) \) (or \( S(x) \)), we have \( \delta(x) = 0 \) (or \( \delta'(x) = 0 \)).

For the example of §4 we have

\[ \delta(\omega) = \omega^2 - \omega x_1, \quad \delta'(\omega) = \omega^2 - 2\omega x_1 + x_1^2. \]

Hence \( \delta(x) = 0 \), while \( \delta'(x) = x_1^2 - x_1 e_1 - x_1 x_2 e_2. \)

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ON THE NUMBER \((10^{23} - 1)/9\)

D. H. Lehmer

The purpose of this note is to save any further effort* in trying to factor the number \( N = (10^{23} - 1)/9 = 111, 11111, 11111, 11111, 11111 \) which in a previous paper was found to be composite.† This assertion was based on a negative result giving \( 3^{N-1} \equiv 1 \pmod{N} \).

On the basis of this conclusion Kraitchik‡ attempted to factor \( N \) arriving at another negative result that \( N \) had no factors and therefore was a prime. This conflict of results led us to recompute the value of \( 3^{N-1} \pmod{N} \) which shows clearly a mistake in the original calculation arising from the choice of 3 for a base instead of another number prime to \( 10^{23} - 1 \). Such another base would have furnished the extra check which would have detected the error.

* A recent letter from Mr. R. E. Powers informs us that he has been to the trouble of finding 150 quadratic residues of \( N \).


The recomputation revealed the following results:

\[
3^{N-1} \equiv 1 \pmod{N},
\]
\[
3^{(N-1)/11} \equiv 1445009647877186725049 = r_1 \pmod{N},
\]
\[
3^{(N-1)/4093} \equiv 9837816775637376837434 = r_2 \pmod{N},
\]
\[
((r_1 - 1), N) = ((r_2 - 1), N) = 1.
\]

By Theorem 3 of my paper cited above, it follows that the factors of \(N\) belong to the forms

\[
\begin{align*}
23n + 1 \\
121n + 1 \\
11390819n + 1 \\
4093n + 1
\end{align*}
\]

If we seek to express \(N\) as the difference of squares \((a^2 - b^2)\), we have

\[
a = 129750757490761n + 115222895547343.
\]

If we restrict \(a\) modulo 12 and 25, the smallest admissible value to try is

\[
a = 5435003952668544.
\]

The total range for \(a\) is given by the inequalities

\[
N^{1/2} < a < \frac{1}{2} \left( W + \frac{N}{W} \right),
\]

where \(W = 22781638\), that is,

\[
a < 243861122499491.
\]

The maximum value of \(a\) is less than the smallest possible value; therefore \(a\) does not exist and \(N\) is a prime.

The results of Kraitchik’s investigations will occupy a whole chapter of his forthcoming book.* Those interested in the factorization of large numbers will await with interest the exposition of the method by which Kraitchik was able to identify this sixth largest prime known.

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