NOTE ON LINEAR TRANSFORMATIONS OF
n-ICS IN m VARIABLES*

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Let us consider the n-ic in m variables

\[ F(x_1, x_2, \cdots, x_m) = 0. \]  

If we subject (1) to the linear transformation

\[ \rho x_1 = a_{11} x'_1 + a_{12} x'_2 + a_{13} x'_3 + \cdots + a_{1m} x'_m, \]
\[ \rho x_2 = a_{21} x'_1 + a_{22} x'_2 + \cdots + a_{2m} x'_m, \cdots, \]
\[ \rho x_m = a_{m1} x'_1 + a_{m2} x'_2 + a_{m3} x'_3 + \cdots + a_{mm} x'_m, \]

we obtain

\[ F(a_{11} x'_1 + a_{12} x'_2 + a_{13} x'_3 + \cdots + a_{1m} x'_m, a_{21} x'_1 + a_{22} x'_2 + \cdots + a_{2m} x'_m, \cdots, a_{m1} x'_1 + a_{m2} x'_2 + \cdots + a_{mm} x'_m) = 0. \]

Note that in the expansion of (3) the coefficient of the term in \( x'_i \), \((i = 1, 2, 3, \cdots, m)\), is \( F(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi}) \). A necessary and sufficient condition for this coefficient to vanish is that the point \( P_i(a_{1i}, a_{2i}, \cdots, a_{mi}) \) shall lie on the geometric locus of (1). To obtain the coefficient of such a term as \( x'_i x'_j \) in the expansion of (3) we can put

\[ x'_i = x'_j = x'_3 = \cdots = x'_{i-1} = x'_{i+1} = \cdots = x'_m = 0, \]

then use Taylor’s Expansion on

\[ F(a_{1i} x'_i, a_{2i} x'_i, a_{3i} x'_i, \cdots, a_{mi} x'_i, a_{mj} x'_j) = F(X_1 + x'_{i1}, X_2 + x'_{i2}, X_3 + x'_{i3}, \cdots, X_m + x'_{im}), \]

* Presented to the Society, August 29, 1929.
where $X_1 = a_1 x_1', X_2 = a_1 x_2', X_2 = a_2 x_3', X_2 = a_2 x_4'$, etc. We find the coefficient of $x_1' x_2'^{n-r}$ in the group of terms\(^*\)

$$\frac{1}{(n-r)!} \left( \frac{\partial F}{\partial X_1} X_1' + \frac{\partial F}{\partial X_2} X_2' + \frac{\partial F}{\partial X_3} X_3' + \cdots + \frac{\partial F}{\partial X_m} X_m' \right)^{(n-r)}$$

\[(5)\]

$$\equiv \left( \frac{\partial F}{\partial a_{ij}} a_{ij} + \frac{\partial F}{\partial a_{2i}} a_{2i} + \cdots + \frac{\partial F}{\partial a_{mi}} a_{mi} \right)^{(n-r)},$$

where $\frac{\partial F}{\partial a_{ij}}$ means $\frac{\partial F}{\partial X_1}$ with $X_1$ replaced by $a_{11}$, $X_2$ by $a_{21}$, $X_3$ by $a_{31}$, $X_m$ by $a_{m1}$, and similarly for $\frac{\partial F}{\partial a_{2i}}$, $\frac{\partial F}{\partial a_{3i}}$, $\cdots$, $\frac{\partial F}{\partial a_{mi}}$. Hence we may conclude that a necessary and sufficient condition for the vanishing of the coefficient of $x_1' x_2'^{n-r}$ in the expansion of $(3)$ is that the point $P_j(a_{1j}, a_{2j}, a_{3j}, \cdots, a_{mj})$ shall lie on the $(n-r)$th polar of $P_i(a_{1i}, a_{2i}, a_{3i}, \cdots, a_{mi})$ with respect to the locus of $(1)$.

We obtain the coefficient of the term in $x_1' x_2'^{r+r'} x_3'^{t} \cdots x_p'^{u+v}$ (where $r+s+t+\cdots+u+v=n$) in the expansion of $(3)$ by the following device. We can write $(3)$ as

$$F(Y_1 + Y_1', Y_2 + Y_2', Y_3 + Y_3', \cdots, Y_m + Y_m') = 0,$$

where

\[
\begin{align*}
Y_1 &= a_{11} x_1' + a_{11} x_1' + a_{12} x_2' + a_{13} x_3' + \cdots + a_{1l-1} x_{l-1}' + a_{1l+1} x_{l+1}' + a_{1l+2} x_{l+2}' + \cdots + a_{1m} x_m' , \\
Y_2 &= a_{21} x_1' + a_{21} x_1' + a_{22} x_2' + a_{23} x_3' + \cdots + a_{2l-1} x_{l-1}' + a_{2l} x_{l}' + a_{2l+1} x_{l+1}' + \cdots + a_{2m} x_m' ,
\end{align*}
\]

etc. We take the collection of terms

$$\frac{1}{(n-r)!} \left( \frac{\partial F}{\partial Y_1} Y_1' + \frac{\partial F}{\partial Y_2} Y_2' + \frac{\partial F}{\partial Y_3} Y_3' + \cdots + \frac{\partial F}{\partial Y_m} Y_m' \right)^{(n-r)}$$

\[(7)\]

* See Goursat-Hedrick *Mathematical Analysis*, vol. 1, pp. 107-108. For the Galois fields, see A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361-363. The methods of this paper may be readily generalized to the polars of an $n$-ic in $m$ variables.
in the expansion of (6). All the terms with \( x^r_l \) as a factor must come from (7). We can write (7) in the form

\[
\frac{x^r_l}{(n-r)!} \left( \frac{\partial F'}{\partial a_{3i}} Y^r_i + \frac{\partial F'}{\partial a_{2i}} Y^r_2 + \frac{\partial F'}{\partial a_{3i}} Y^r_3 + \cdots + \frac{\partial F'}{\partial a_{m_i}} Y^r_m \right)^{(n-r)}.
\]

If we equate (8) to zero we obtain the \((n-r)\)th polar of \( P_t(a_{1i}, a_{2i}, a_{3i}, \ldots, a_{m_i}) \) with respect to the locus of (1). We can also write (7) in the form

\[
\frac{x^r_l}{(n-r)!} F'(Y^r_1, Y^r_2, Y^r_3, \ldots, Y^r_m),
\]

where \( F' \) is a function of the \((n-r)\)th degree. We put

\[
Y^r_1 = Z_1 + Z'_1, Y^r_2 = Z_2 + Z'_2, \ldots, Y^r_m = Z_m + Z'_m,
\]

where

\[
Z_1 = a_{1i} x^r_l, Z'_1 = \sum a_{1i} x' + a_{13} x'_3 + \cdots + a_{1i-1} x'_{i-1}
\]
\[
+ a_{1i+1} x'_{i+1} + a_{1i+2} x'_{i+2} + \cdots + a_{1j-1} x'_{j-1} + a_{1j+1} x'_{j+1}
\]
\[
+ a_{1j+2} x'_{j+2} + \cdots + a_{1m} x'_m, Z_2 = a_{2i} x^r_l,
\]

etc. Expanding (9), we find that all the terms in the expansion of (3) that have the factors \( x^r_l \) and \( x^r_j \) must be in the collection of terms

\[
\frac{x^r_l}{(n-r)!((n-r)-(r-s))!} \left( \frac{\partial F'}{\partial Z_1} Z'_1 + \frac{\partial F'}{\partial Z_2} Z'_2 + \frac{\partial F'}{\partial Z_3} Z'_3 + \cdots + \frac{\partial F'}{\partial Z_m} Z'_m \right)^{(n-r)}
\]

\[
+ \left( \frac{\partial F'}{\partial a_{2i}} Z^r_2 + \frac{\partial F'}{\partial a_{3i}} Z^r_3 + \cdots + \frac{\partial F'}{\partial a_{m_i}} Z^r_m \right)^{(n-r)}
\]

If we equate (11) to zero we shall have the \((n-r-s)\)th polar of the point \( P_t(a_{1i}, a_{2i}, a_{3i}, \ldots, a_{m_i}) \) with respect to the \((n-r)\)th polar of \( P_t(a_{1i}, a_{2i}, a_{3i}, \ldots, a_{m_i}) \) with respect to the locus of (1).
Next we take
\[ Z'_1 = W'_1 + W'_2, \quad Z'_2 = W'_2 + W'_3, \]
\[ Z'_3 = W'_3 + \cdots, \quad Z'_m = W'_m + W'_m \]
in (11), where
\[ W_1 = a_{1k}x'_k, \quad W'_1 = a_{11}x'_1 + a_{12}x'_2 + \cdots + a_{1i-1}x'_{i-1} \]
\[ + a_{1i+1}x'_{i+1} + \cdots + a_{1j-1}x'_{j-1} + a_{1j+1}x'_{j+1} + \cdots \]
\[ + a_{k-1}x'_{k-1} + a_{k+1}x'_{k+1} + \cdots + a_{m}x'_m, \quad W'_2 = a_{2k}x'_k, \]
etc. We repeat the above processes until we finally reach the collection of terms having all the factors \( x'_1, x'_2, x'_3, \cdots, x'_u, \) and \( x'_v. \) Therefore, we see that for the coefficient of the term \( x'_1 \cdots x'_m \) in the expansion of (3) to vanish we must have the point \( P_\pi(a_1\pi, a_2\pi, a_3\pi, \cdots, a_m\pi) \) on the \((n-r-s-t-u-v)\)th polar of \( P_1(a_{11}, a_{21}, \cdots, a_{m1}) \) with respect to the \( \cdots \)th polar of \( \cdots, \cdots, \cdots, \) with respect to the \((n-r-s-t)\)th polar of \( P_k(a_{1k}, a_{2k}, \cdots, a_{mk}) \) with respect to the \((n-r-s)\)th polar of \( P_j(a_{1j}, a_{2j}, \cdots, a_{mj}) \) with respect to the \((n-r)\)th polar of \( P_i(a_{1i}, a_{2i}, \cdots, a_{mi}) \) with respect to the locus of (1).

It is noteworthy that this discussion applies to the ordinary complex or real domains and also to the Galois fields.

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