1. Exceptional Occurrence of Singular Manifolds. This note is concerned with the following theorem.

Let \( R \) denote a closed region, consisting of an open continuum of the space of the \( n \) complex variables \( z_1, z_2, \ldots, z_n \), together with its boundary points. Let the functions \( F_i(z_1, z_2, \ldots, z_n) \), \( i = 1, 2, \ldots, m \), be analytic at all points of \( R \). Let \( M_k \) denote the matrix

\[
(M_k) = \begin{pmatrix}
\frac{\partial F_i}{\partial z_j} \\
\end{pmatrix}, \quad (i = 1, 2, \ldots, k), \quad (j = 1, 2, \ldots, n).
\]

We assume that \( M_m \) is of rank \( m \) at some point of \( R \).

Consider the manifold defined by the equations

\[
F_1(z_1, z_2, \ldots, z_n) = c_1, \ldots, F_m(z_1, z_2, \ldots, z_n) = c_m,
\]

where \( c_1, c_2, \ldots, c_m \) are complex constants.

For all but a finite number of values of \( c_1 \) the manifold defined by the first equation \( A \) contains no points in \( R \) at which the rank of \( M_1 \) is less than \( 1 \).

If \( c_1, c_2, \ldots, c_k \) have been chosen so that the matrix \( M_k \) is of rank \( k \) at every point in \( R \) on the manifold defined by the first \( k \) equations \( A \), then for all but a finite number of values of \( c_{k+1} \) the manifold defined by the first \( k+1 \) equations \( A \) contains no points in \( R \) at which the rank of the matrix \( M_{k+1} \) is less than \( k+1 \).

Thus, if \( c_1, c_2, \ldots, c_m \) are chosen in order, each avoiding a certain finite set of values, the manifold defined by the equations \( A \) will have no singular points in \( R \).\[ •

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† As far as I know, a proof of this general theorem has not been published. Birkhoff and I gave it for the case in which the functions \( F_i \) are polynomials (Transactions of this Society, vol. 23 (1922), pp. 97–98), and
By a singular point of a manifold, we mean a point at which a unique tangent linear manifold of the same order does not exist. A sufficient condition that a point of the manifold defined by the equations $(A)$ be not singular is that the rank of the matrix $M_m$ is $m$ at that point. That the condition is not necessary may be seen by the example

$$z_1^2z_2^2 - z_3^2 = 0, \quad z_1^2z_2^2 + z_3^2 = 0.$$  

The manifold defined by these equations consists of the two intersecting straight lines $z_1 = z_2 = 0$ and $z_1 = z_3 = 0$. Its only singular point is at the origin, yet the rank of $M_2$ is less than 2 at every point of the manifold.

It is not stated in the theorem that the values of the $c_j$ characterizing singular manifolds are finite in number. In general, the value of $c_{k+1}$ to be avoided, if the resulting manifold is to be non-singular, is a function of the $c_j$ previously chosen. Thus in the case

$$z_1 - z_2z_3 = c_1, \quad z_1 = c_2,$$

the value of $c_2$ to be avoided is $c_1$. What is the case, is that among the $\infty^{2m}$ members of the family defined by $(A)$, there are at most $\infty^{2m-2}$ singular manifolds, a precise interpretation of this statement being given in the theorem.

2. Needed Results in the Theory of Analytic Functions. We attach our considerations to the study of simultaneous equations in the second volume of Osgood’s *Lehrbuch der Funktionentheorie*, where we find a theorem, of which the part we need may be stated as follows:

**Given a system of simultaneous equations**

$$G_1(z_1, z_2, \ldots, z_n) = 0, \ldots, G_l(z_1, z_2, \ldots, z_n) = 0,$$

where $G_k(z_1, z_2, \ldots, z_n), \; k=1, 2, \ldots, l$, is analytic at the point $(a) = (a_1, a_2, \ldots, a_n)$ and vanishes there, but does not

Morse gives a proof for the case $m=1$ in a paper in the April number of the American Journal of Mathematics, vol. 51 (1929). The theorem is apparently useful; in addition to its role in the two papers just cited it has served me at several points in a forthcoming book on Potential Theory.  

vanish identically. The solutions of this system in the neigh-
borhood of (a) consist either of the point (a) alone, or of one or
more manifolds g. The context makes it clear that the number
of these manifolds is finite.

The essential property of the manifolds g for our purposes
is that any two points of any one of them can be connected by a
curve γ, lying in the manifold, and analytic except at a finite
number of points.

We begin, in establishing this property, with a pseudo-
algebraic manifold ∅ defined by the irreducible equation

\[ H(w, u_1, u_2, \ldots, u_p) = w^r + E_1 w^{r-1} + \cdots + E_r = 0 \]

and the inequalities \(|u_i| < h\), where \(h\) is a suitably restricted
positive number. The coefficients \(E_j\) are power series in the
variables \(u_i\) convergent for \(|u_i| < h', h' > h\), vanishing at the
origin \(O(u_1 = u_2 = \cdots = u_p = 0)\).

If \(P'(w', u'_1, u'_2, \ldots, u'_p)\) is any point of ∅, and if in the
equation (1) we set \(u_1 = u'_1 t, u_2 = u'_2 t, \ldots, u_p = u'_p t\), the re-
sulting equation defines \(w\) as an algebroid function of \(t\). If
the latter equation is reducible, we select the irreducible
factor of the left hand member which vanishes for \(w = w'\)
when \(t = 1\), and equate it to 0. The discriminant of the result-
ing equation vanishes at most a finite number of times on the
real interval \((0, 1)\), and hence from the branches of the
function \(w\) of \(t\) there can be selected a (complex) single-
valued function \(w = f(t)\), which is continuous on the closed
interval \((0, 1)\), analytic except at a finite number of points,
and which reduces to 0 for \(t = 0\), and to \(w'\) for \(t = 1\). Then the
curve

\[ (\gamma') \quad w = f(t), \quad u_1 = u'_1 t, \ldots, \quad u_p = u'_p t, \quad 0 \leq t \leq 1, \]

lies in ∅, connects \(P'\) with \(O\), and is analytic except at a
finite number of points. By means of a curve consisting of
two such parts, any two points \(P'\) and \(P''\) of ∅ can be
connected.

We now consider a set of functions \(w_1, w_2, \ldots, w_s\) of
$u_1, u_2, \cdots, u_p$, belonging to the pseudo-algebraic manifold $\mathfrak{S}$, that is, having the following properties: they are one-valued and continuous on $\mathfrak{S}$, and analytic at every point of $\mathfrak{S}$ at which the discriminant of the equation (1) does not vanish; the function $\alpha_1 w_1 + \cdots + \alpha_p w_p$ of the $u_i$ has, for non-specialized values of the parameters $\alpha_1, \cdots, \alpha_p$, branches, corresponding to the branches of $w$, no two of which are identically equal. Then the set of points of the space of $(w_1, w_2, \cdots, w_p, u_1, u_2, \cdots, u_p)$ which correspond to the points $(w, u_1, \cdots, u_p)$ of $\mathfrak{S}$ constitutes a manifold $\mathfrak{g}$. The most general manifold $\mathfrak{g}$ is either one defined in this way, or else one derived from it by a non-singular linear transformation of the variables. As such a transformation does not change the essential character of the curve $\gamma$, we may confine ourselves to the manifolds $\mathfrak{g}$ as first defined.

The functions $w_1, w_2, \cdots, w_p$ are pseudo-algebraic, and as they are one-valued and continuous on $\mathfrak{S}$, they assume at $\mathfrak{O}$ unique values $w_{10}, w_{20}, \cdots, w_{p0}$. If $p'(w_1', w_2', \cdots, w_p', u_1', u_2', \cdots, u_p')$ is any point of $\mathfrak{g}$, there corresponds to it a unique point $\mathfrak{P}'$ of $\mathfrak{S}$, which is connected with $\mathfrak{O}$ by a curve $\gamma'$. Just as the function $w = f(t)$ was determined, we may determine from among the branches of $w_1$, a single-valued function $w_1 = f_1(t)$, continuous on the interval $(0, 1)$, analytic, except at a finite number of points, and reducing to $w_{10}$ for $t = 0$ and to $w_1'$ for $t = 1$. When this and similar functions have been determined for $w_1, \cdots, w_p$, we shall have a curve

$\gamma: w_1 = f_1(t), \cdots, w_p = f_p(t), u_1 = u_1't, \cdots, u_p = u_p't, \quad 0 \leq t \leq 1,$

from two of which can be constructed a curve of the required character, connecting any two given points of $\mathfrak{g}$.

3. Proof of the Theorem. Let $(a)$ be any point of $\mathfrak{R}$. Then in the neighborhood of $(a)$, the solutions of the equations

$\frac{\partial F_1}{\partial z_1} = 0, \quad \frac{\partial F_1}{\partial z_2} = 0, \cdots, \frac{\partial F_1}{\partial z_n} = 0,$
(not all of which are identically satisfied, since $M_m$ is of rank $m$ at some point of $R$), if there are any, consist, by Osgood's theorem, either of the point $(a)$ alone, or of a finite number of manifolds $g$. Along an analytic arc of a curve $\gamma$ joining any two points of such a manifold, $dF_1=0$. It follows that $F_1$ is constant on each such manifold, so that the number of values of $F_1$ corresponding to solutions of the equations (2) in the neighborhood of $(a)$, is finite. By the Heine-Borel theorem, the same is true for the whole region $R$. Hence for all but a finite number of values of $c_1$ the manifold defined by the first equation $(A)$ contains no points in $R$ at which $M_1$ is of rank less than 1. The first part of the theorem is thus established.

Suppose now that $c_1, c_2, \cdots, c_k$ have been chosen so that at every point in $R$ of the manifold defined by the first $k$ equations $(A)$, the matrix $M_k$ is of rank $k$. We consider the simultaneous system of equations consisting of the two sets

(3) $F_1(z_1, z_2, \cdots, z_n) - c_1 = 0, \cdots, F_k(z_1, z_2, \cdots, z_n) - c_k = 0,$

and

(4) $D_1(z_1, z_2, \cdots, z_n) = 0, \cdots, D_r(z_1, z_2, \cdots, z_n) = 0,$

where $D_1, \cdots, D_r$ are all the determinants of order $k+1$ formed from the matrix $M_{k+1}$, except such as may be identically 0. Not all vanish identically, since, by hypothesis, $M_m$ is of rank $m$ at some point of $R$.

If this system has any solutions in the neighborhood of a point $(a)$ of $R$, they consist, by Osgood's theorem, of the point $(a)$ alone, or of a finite number of manifolds $g$. Let $g$ denote such a manifold. On it $M_k$ and $M_{k+1}$ are everywhere of rank $k$. Hence from the equations $dF_1 = dF_2 = \cdots = dF_k = 0$, which follow from (3), we infer that $dF_{k+1} = 0$ at all points of the curve $\gamma$ connecting any two given points of $g$. By reasoning previously employed, it follows that for all but a finite number of values of $c_{k+1}$, the matrix $M_{k+1}$ is of rank $k+1$ at all points of $R$ on the manifold defined by the first $k+1$ equations $(A)$. This proves the second part of the theorem.
The truth of the final statement of the theorem emerges when to $k$ are assigned successively the values $1, 2, \ldots, m-1$, and when it is recalled that a sufficient condition that the manifold defined by the equations $(A)$ has no singular points in $R$ is that the matrix $M_m$ is of rank $m$ at all points of the manifold in $R$.

Harvard University

ON THE FOUNDATIONS OF GENERAL INFINITESIMAL GEOMETRY*

BY HERMANN WEYL

In connection with a seminar on infinitesimal geometry in Princeton, in which I took part, it seemed desirable to clarify the relations between the work of the Princeton school and that of Cartan.

With a group $\mathcal{O}$ of transformations in $m$ variables $\xi$ is associated, in accordance with Klein's Erlanger Program, a homogeneous or plane space $\mathcal{R}$ of the kind $\mathcal{O}$; a point of $\mathcal{R}$ is represented by a set of values of the "coordinates" $\xi^a$ and figures which go into each other on subjecting the coordinates to a transformation of $\mathcal{O}$ are to be considered as fully equivalent. The transformations of $\mathcal{O}$ give at the same time the transition between two allowable "normal" coordinate systems in $\mathcal{R}$. If we have two spaces $\mathcal{R}, \mathcal{R}'$ of the kind $\mathcal{O}$ and set up a definite normal coordinate system in each of them, then such a transformation can be interpreted as an isomorphic representation of $\mathcal{R}$ on $\mathcal{R}'$. $\mathcal{O}$ is assumed to be transitive.

Cartan† developed a general scheme of infinitesimal geometry in which Klein's notions were applied to the tangent plane and not to the $n$-dimensional manifold $M$ itself. The

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