ON THE NATURE OF $\theta$ IN THE MEAN-VALUE THEOREM OF THE DIFFERENTIAL CALCULUS*

BY GANESH PRASAD

1. Introduction. If $f(x)$ is a single-valued function which is finite and continuous in an interval $(a, b)$, the ends being included, than the relation

\[(M)\quad f(x + h) = f(x) + hf'(x + \theta h), \quad 0 < \theta < 1,\]

holds for every value of $x$ and $h$ for which the interval $(x, x+h)$ is in the interval $(a, b)$; provided that either $f'(x)$ exists at every point inside the interval $(a, b)$ or a certain less restrictive condition† is satisfied. In recent years the nature of $\theta$ has been studied by a number of writers‡ who start with the assumption that $f''(x)$ exists everywhere in the interval $(a, b)$. The two theorems, which it is the object of this paper to formulate and prove, are believed to be new and hold even if $f''(x)$ does not exist everywhere. For the sake of clarity and fixity of ideas, I consider $\theta$ only as a function of $h$, assuming $x$ to be a constant, say 0, in the theorem (M).

2. Theorem I. If $\theta(h)$ is single-valued and continuous, it is not necessarily differentiable for every value of $h$.

Proof. Take $f(x)$ to be the indefinite integral of a monotone, increasing and continuous function which has a differential coefficient everywhere in the interval $(a, b)$, excepting the points

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* Presented to the Society, February 25, 1928.
of an everywhere dense set. Such a function is that given by T. Broden.* Denoting Broden's function by $\varphi$, let

$$ f(x) = \int_0^x \varphi(t) \, dt, $$

and let the everywhere dense set be denoted by $S$; also let $\xi$ stand for $h\theta$. Then it is easily seen that $\xi$ is a single-valued and continuous function of $h$, and that, corresponding to each value of $\xi$, there is a value of $h$ and only one value. Now (M) gives

$$ f(h) = h\varphi'(\xi) = h\varphi(\xi), $$

whatever $h$ may be. Therefore, as $f'(h)$ exists,

$$ \frac{d}{dh} \left\{ h\varphi(\xi) \right\}, \quad \text{that is,} \quad \varphi(\xi) + h \frac{d\varphi}{dh}, $$

must exist for every value of $h$. Thus, at any point $h = h'$ which corresponds to a point $\xi = \xi'$ of $S$, $d\xi/dh$ and, consequently, $d\theta/dh$ must be non-existent; otherwise $\varphi'(\xi')$ will exist which is impossible.

Therefore it is proved that, for every value of $h$ corresponding to which $\xi$ is a point of $S$, $d\theta/dh$ is non-existent.

3. Theorem II. If $\varphi(h)$ is single-valued, it is necessarily continuous for every value of $h$.

Proof. Assume, if possible, that $h$ is a point of discontinuity of $\varphi(h)$. Then, denoting the corresponding values of $\xi$ and $\theta$ by $\bar{\xi}$ and $\bar{\theta}$ respectively, we have by (M)

$$ f(h) = \bar{h}\varphi'(\bar{\xi}). $$

Now two possibilities arise: the discontinuity may be of the first kind or of the second kind.

(a) If the discontinuity is of the first kind, then there must be a sequence $\{h_n\}$, tending to $\bar{h}$, for which the corresponding sequence $\{\xi_n\}$ does not tend to $\bar{\xi}$ but to $\bar{\xi}'$ different from $\bar{\xi}$. Thus

$$ f'(\bar{\xi}) = f'(\bar{\xi}'). $$

So, for the same value of \( h \), namely, \( \bar{h} \), there are two values of \( \theta \), namely, \( \bar{\theta} \) and \( \theta' \), which is absurd, since \( \theta \) is single-valued.

(b) If the discontinuity is of the second kind, then there must be a sequence \( \{h_n\} \), tending to \( \bar{h} \), for which the corresponding sequence \( \{\xi_n\} \) does not tend to any limit. Therefore two values \( k_1 \) and \( k_2 \) of \( h \) can always be found as near as we please to \( \bar{h} \) such that the corresponding values \( \eta_1 \) and \( \eta_2 \) of \( \xi \) differ from each other by a quantity greater than a suitably prescribed positive quantity \( \delta \). But, from (M), \( f(h)/h \) and, consequently, \( f'(\xi) \) are continuous functions of \( h \) at \( \bar{h} \). Therefore \( \xi \) must be multiple-valued at \( \bar{h} \), which is absurd, since \( \theta \) is single-valued.

The University of Calcutta

A Numerical Function Applied to Cyclotomy

By Emma T. Lehmer

A function \( \phi_2(n) \) giving the number of pairs of consecutive integers each less than \( n \) and prime to \( n \), was considered first by Schemmel.* In applying this function to the enumeration of magic squares, D. N. Lehmer† has shown that if one replaces consecutive pairs by pairs of integers having a fixed difference \( \lambda \) prime to \( n = \prod_{i=1}^{t} p_i^{e_i} \), then the number of such pairs (mod \( n \)) whose elements are both prime to \( n \) is also given by

\[
\phi_2(n) = \prod_{i=1}^{t} p_i^{e_i-1}(p_i - 2) .
\]

As is the case for Euler's totient function \( \phi(n) \), the function \( \phi_2(n) \) obviously enjoys the multiplicative property \( \phi_2(m)\phi_2(n) = \phi_2(mn) \), \( (m, n) = 1 \), \( \phi_2(1) = 1 \). In what follows we call an integer simple if it contains no square factor \( >1 \). For a simple number \( n \) we have the following analog of Gauss' theorem:

\[
(1) \quad \sum_{d|n} \phi_2(d) = \phi(n) ,
\]

† Transactions of this Society, vol. 31 (1929), pp. 538–9.