ON A COMPLETE CHARACTERIZATION OF THE SET OF POINTS OF UNBOUNDED GRADE OF AN ARBITRARY SURFACE*

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Let \( z = f(x, y) \) be an arbitrary surface \( S \), in the sense that \( f(x, y) \) is an arbitrary one-valued real function of the real variables \( x \) and \( y \). By the grade of a segment joining two points \( A \) and \( B \), we understand the absolute value of the tangent of the angle which \( AB \) makes with the \( xy \)-plane. The point \( A = (\xi, \eta, \xi) \) of the surface \( S \) is said to be of bounded grade—or \( S \) is said to be of bounded grade at the point \( (\xi, \eta) \)—if the grade of \( AB \) is bounded for all \( B = (x, y, z) \) of \( S \) at a sufficiently small “horizontal” distance \( [(x-\xi)^2+(y-\eta)^2]^{1/2} \) from \( A \). If \( A \) does not satisfy this condition, \( S \) is said to be of unbounded grade at \( (\xi, \eta) \). It is the object of the present paper to prove the following theorem, which identifies the aggregate—for the totality of arbitrary surfaces—of sets of points of unbounded grade with the aggregate of sets of type \( G_6 \).†

**Theorem.**† The set of points \( (x, y) \) at which an arbitrary surface \( z = f(x, y) \) is of unbounded grade is a \( G_6 \). Conversely if a \( G_6 \) is given, there exists a surface \( z = f(x, y) \) such that this \( G_6 \) is identical with the set of points \( (x, y) \) where the surface is of unbounded grade.

**Proof.** If \( P = (x, y) \) is a point at which the given surface \( S \), represented by \( z = f(x, y) \), is of unbounded grade, we may, for every positive integer \( n \), find a point \( P_n = (x_n, y_n) \) such that the distance \( d(PP_n) \) between \( P \) and \( P_n \) is less than \( 1/n \), and \( g(f, PP_n) > n \), understanding by \( g(f, PP_n) \) the grade of the segment joining the points of \( S \) corresponding to \( P \) and \( P_n \). Enclose \( P \) and \( P_n \) in a circle \( C_{P_n} \), regarded as made up only of

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† A \( G_6 \) is the product of \( \mathbb{R} \) open sets. The notation is due to Hausdorff.
‡ The direct part of this theorem is stated by W. H. Young for the case of a function of one variable; see Arkiv för Matematik, Astronomi och Fysik, vol. 1 (1903).
interior points, of diameter $< 2d(PP_n)$. Let $G_n$ equal the sum of the $C^0_P$ for $n$ fixed and $P$ ranging over the set $U$ of points at which $S$ is of unbounded grade; and let $T = \prod G_n$. We show that $T = U$. Of course, since every point of $U$ is in $G_n$ for every $n$, we have $U \subseteq T$. Suppose $Q$ is in $C^0_P$. Then either $|f(Q) - f(P)|$ or $|f(P) - f(P_n)|$, where $f(P) = f(x, y)$, with similar meaning for $f(Q)$ and $f(P_n)$, is not less than $\frac{1}{2} |f(P) - f(P_n)|$. Since $d(PP_n)$ and $d(QP_n)$ are both less than $2d(PP_n)$, we conclude that either $g(f, QP)$ or $g(f, QP_n) > g(f, PP_n)/4 > n/4$. It follows that if $Q$ is in some $C^0_P$ for every $n$, then $S$ is of unbounded grade at $Q$. Hence $T \subseteq U$, and therefore $T = U$.

To prove the converse part of the theorem, we suppose that $\prod G_n$ is a given product of open sets $G_n$ lying in the $xy$-plane. Let $G^{(n)} = \prod G_n$, and $F_n =$ the complement of $G^{(n)}$ with respect to the $xy$-plane. In terms of these $F_n$, we shall define the surfaces $z = f_n(x, y)$; and $z = f(x, y) = \sum f_n(x, y)$ will be the required surface having bounded grade at the points of $\sum F_n$ and unbounded grade at the points of $\prod G_n$.

To this end, we suppose that $T_n = \{Q^{(n)}\}$ is, for every positive integer $n$, a system of non-overlapping squares $Q^{(n)}$ lying in $G^{(n)}$, such that every point of $G^{(n)}$ is in the interior or on the boundary of at least one $Q^{(n)}$ of $T_n$. We define $f_n(x, y)$ as $0$ at the points of $F_n$ and at the boundary points of the $Q^{(n)}$. If $P = (x, y)$ is an interior point of the square $Q^{(n)}$, we set

$$f_n(P) = f_n(x, y) = \rho_n d_P^{(n)},$$

where $d_P^{(n)}$ is the distance from $P$ to the boundary of $Q^{(n)}$, and $\rho_n$ is a number, depending on $n$ but not on the varying $Q^{(n)}$ of $T_n$, and subject to certain relations to be stated presently. We suppose furthermore that $T_{n+1}$ is a “subdivision” of $T_n \{n = 1, 2, \ldots\}$ in the sense that every $Q^{(n+1)}$ of $T_{n+1} \{n = 1, 2, \ldots\}$ lies in just one $Q^{(n)}$ of $T_n$. Let $2q^{(n)}$, which may vary, as $Q^{(n)}$, $n$ fixed, varies, be the length of side of $Q^{(n)}$. Then, as we may, we select the $\rho_n$ and $q^{(n)}$ in such a way that

(a) $\rho_n / \sqrt{2} > M_{n-1} + n(n \geq 2); \quad \rho_1 = 1;

(b) $\rho_n q^{(n)} < d_P^{(n)}/2^n;

(c) $\rho_n q^{(n)} < q^{(n-1)}$.

Here $M_n$ represents the upper boundary, which is evidently
finite, of the grade of a segment with end points on the surface

\[ z = s_n(x, y) = \sum f_n(x, y), \]

for example, \( M_1 = 1 \); and \( d_n \), which depends, for \( n \) fixed, on \( Q^{(n)} \), is the minimum distance from the boundary points of \( Q^{(n)} \) to \( F_n \); moreover, inequality (c) is to be understood as demanded only in case the square \( Q^{(n)} \) of side \( 2q^{(n)} \) lies in, or has at least one boundary point in common with the square \( Q^{(n-1)} \) of side \( 2q^{(n-1)} \). We now set \( f(x, y) = \sum f_n(x, y) \), and observe that \( f \) exists. For, by the definition of \( f_n \) and inequality (b), if \( P \) is an interior point of some \( Q^{(n)} \) of \( T_n \), then

\[ f_n(P) = r_n d_n \leq r_n q^{(n)} < d_n^2/2^n, \]

where \( d_n \) is the minimum distance from \( P \) to \( F_n \). Since \( d_n \) does not increase with \( n \), and \( f_n(P) = 0 \) if \( P \) is interior to no \( Q^{(n)} \) of \( T_n \), it follows that \( \sum f_n(P) \) is convergent. We shall now prove that \( f \) has the required properties.

First, let \( P \) be a point of \( F_n \). If \( n > n \), and \( P' \) is a point of \( G \), lying in the interior of the square \( Q^{(r)} \) of \( T_r \), we have, by inequality (b),

\[ g(f_n, PP') \leq \frac{r_n q^{(n)}}{d(PP')} \leq \frac{r_n q^{(n)}}{d_r} < \frac{d(PP')}{2^r}. \]

If \( P' \neq P \) is interior to no \( Q^{(r)} \) of \( T_r \), \( f_n(P') = 0 \), and since \( f_n(P) = 0 \), we have \( g(f_n, PP') = 0 \). It follows, if \( r_n(x, y) = \sum f_n(x, y) \), that \( g(r_n, PP') < d(PP') \) for every point \( P' \neq P \). Since the surface \( z = s_n(x, y) \) is of bounded grade at every point, and

\[ f(x, y) = s_n(x, y) + r_n(x, y), \]

we conclude that the surface \( z = f(x, y) \) is of bounded grade at every point of every \( F_n \), and therefore of bounded grade at every point of \( \sum F_n \).

Now let \( P \) be a point of \( \prod G_n \), and \( Q^{(n)} \) a square of \( T_n \) containing \( P \) in its interior or on its boundary. Then there is a point \( P' \) in \( Q^{(n)} \) such that \( d(PP') \geq q^{(n)}/2 \) and \( g(f_n, PP') \geq r_n/\sqrt{2} \). Since \( g(s_{n-1}, PP') \leq M_{n-1} \), we have, by inequality (a), the relation \( g(s_n, PP') > n(n \geq 2) \). Let \( n \) be an integer greater than \( n \). If \( P \) and \( P' \) both belong to \( G^{(r)} \) there are two squares of \( T_r \), possibly identical, the one containing \( P \) and the other \( P' \); let
$Q^{(v)}$ be the one with the larger (or at least not the smaller) side $q^{(v)}$. Then

$$g(f, PP') \leq \rho q^{(v)}/d(PP') < 2\rho q^{(v)}/q^{(n)}.$$ 

Moreover $Q^{(v)}$ lies in a $Q^{(v-1)}$ and this in turn in a $Q^{(v-2)}$ and so on down to $Q^{(n+1)}$, which lies in or has a boundary point in common with $Q^{(n)}$. Therefore, in virtue of inequality (c),

$$g(f, PP') < \frac{2q^{(v-1)}}{q^{(n)}} = \frac{2q^{(v-1)}}{q^{(v-2)}} \cdots \frac{q^{(n+1)}}{q^{(n)}}$$

$$< \frac{1}{\rho_{v-1}} \cdots \frac{1}{\rho_{n+1}} < \frac{1}{2^{n-2}},$$

since $\rho_n > 2$ for $n > 1$. The reasoning here implies at first that $v > n + 1$, but the final inequality is valid for $v = n + 1$ also, and evidently, too, if either one or both of the points $P$ and $P'$ lie in $F$. Hence

$$g(r_n, PP') < \sum_{r=n+1}^{\infty} \frac{1}{2^{r-n-2}} = 4.$$ 

Therefore, for $n \geq 2$,

$$g(f, PP') \geq g(s_n, PP') - g(r_n, PP') > n - 4.$$ 

Since $P$ belongs to $\prod G_n$, a point $P'$ satisfying the last inequality can be found for every positive integer $n \geq 2$; and since $PP'$, together with $q^{(n)}$ is, by (c), infinitesimal as $n \to \infty$, we conclude that the surface $z = f(x, y)$ is of unbounded grade at $P$.

**Remark.** The proof has been given for a surface $z = f(x, y)$ lying in a euclidean space, but the same reasoning applies to euclidean $n$-space. In fact, with certain modifications of the argument not hard to discern, our theorem, including the converse part, can be extended to any metric space, assumed, of course, if the theorem is to retain significance, to be without isolated points. The triangle postulate $d(P_1P_2) + d(P_2P_3) \geq d(P_1P_3)$ for such a space turns out to be an adequate substitute for metric relations in the plane frequently utilized in the proof. For such an abstract space, we should, for example, change the squares to "spheres"; however, to show that $G^{(n)}$ is the sum of non-overlapping spheres $Q^{(n)}$, boundary included, we make use of Zermelo's Theorem on Normal Order.

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