AN ELEMENTARY THEOREM ON MATRICES*

BY M. H. INGRAHAM

This note applies some elementary algebraic theory to the theory of matrices, securing a generalization of the theorem:

If \( y_1, \ldots, y_n \) are elements of a field \( F \) corresponding to \( n \) distinct elements \( x_1, \ldots, x_n \) in \( F \), there exists one and only one polynomial \( f \) of degree less than \( n \) such that \( f(x_i) = y_i \) \((i = 1, \ldots, n)\).

As is well known, every finite square matrix \( m \) with elements in a field \( F \) satisfies its characteristic equation \( |m - \lambda I| = 0 \), and hence satisfies a unique equation, \( g(\lambda) = 0, \) of minimum degree with leading coefficient unity. Moreover, if for two polynomials \( f \) and \( h, f(m) = h(m), \) then \( f(\lambda) \equiv h(\lambda) \mod g(\lambda) \) and conversely.

We seek an answer to the question “Given finite square matrices \( m_1, \ldots, m_n \) and polynomials \( h_1, \ldots, h_n, \) under what conditions can a polynomial \( f \) be found such that \( f(m_i) = h_i(m_i) \) \((i = 1, \ldots, n)\)?”

If the minimum equations of the \( m_i \) are \( g_i(\lambda) = 0, \) \((i = 1, \ldots, n)\), the above question is equivalent to asking, under what conditions the congruences

\[
f(\lambda) \equiv h_i(\lambda) \mod g_i(\lambda),
\]

\((i = 1, \ldots, n),\)

have a solution.

Standard works on number theory discuss this question and give its solution in an elementary fashion. In particular, there are always solutions when \( g_1, \ldots, g_n \) are relatively prime.

From considerations of this known work on congruences, many theorems, two of which follow, may be translated at once to their matrix form.

**Theorem 1.** If there exists a polynomial \( f \) in a field \( F \) such that for \( n \) finite square matrices \( m_1, \ldots, m_n \) with elements in \( F, \) and \( n \) polynomials \( h_1, \ldots, h_n \) in \( F, f(m_i) = h_i(m_i), \) \((i = 1, \ldots, n),\) then there exists one and only one such \( f \) of degree lower than that of \( k, \) the least common multiple of the minimum equations of \( m_1, \ldots, \)

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THEOREM 2. If the minimum equations of $n$ finite square matrices $m_1$ to $m_n$ with elements in a field $F$ are relatively prime then for any set of $n$ polynomials $h_1, \ldots, h_m$, in $F$, a polynomial $f$ may be found such that

$$f(m_i) = h_i(m_i), \quad (i = 1, \ldots, n).$$

These theorems specialize to the above mentioned algebraic theorem since each $x_i$ is a one by one matrix with minimum equation $\lambda - x_i = 0$.

It should be noted that in the above discussion no restriction as to the field in which the elements of the matrices may lie is made, nor are the $m_i$ necessarily of the same order.

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PROBLEMS OF THE CALCULUS OF VARIATIONS WITH PRESCRIBED TRANSVERSALITY CONDITIONS*

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1. Introduction. Problems of the calculus of variations in the plane for which a prescribed relation exists between the directions of the extremals and the transversals were first studied by Stromquist‡ and Bliss.§ Recently Rawles,|| using a method based on properties of the Hilbert invariant integral, has given an interesting treatment of the analogous problem in $(x, y_1, \ldots, y_n)$-space.

In the present paper the latter problem is attacked from a quite different point of view.¶ The method here used avoids a restrictive hypothesis made by Rawles with regard to the ex-

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|| Rawles, Transactions of this Society, vol. 30 (1928), pp. 765–784.
¶ The possibility of approaching the problem from this viewpoint was suggested to the writer by G. A. Bliss.