ON THE NUMBER OF APPARENT DOUBLE POINTS OF r-SPACE CURVES

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Consider a curve \( C^N \) of order \( N \) in \( r \)-space. The number, \( h \), of \( (r-2) \)-spaces passing through a given \( (r-3) \)-space and meeting \( C^N \) twice is finite. If \( C^N \) is projected on to a 3-space, then \( h \) is the number of apparent double points on the projection. To avoid circumlocution, we shall use the phrase the apparent double points of \( C^N \) instead of the apparent double points of the 3-space projection of \( C^N \).

When the curve \( C^N \) is the intersection of \( r-1 \) hypersurfaces of order \( n_1, n_2, \ldots, n_{r-1} \), the number of its apparent double points is known and is given by the formula*

\[
h = \frac{1}{2}n_1n_2 \cdots n_{r-1}(n_1n_2 \cdots n_{r-1} - \sum n_i + r - 2).
\]

But suppose \( C^N \) is not the intersection of \( r-1 \) hypersurfaces but the intersection of \( q<r-1 \) varieties \( V_{r_1}^n, V_{r_2}^n, \ldots, V_{r_q}^n \) of orders \( n_1, n_2, \ldots, n_q \) and of dimensions (which may be different) \( r_1, r_2, \ldots, r_q \) where

\[
r_1 + r_2 + \cdots + r_q = r(q - 1) + 1.
\]

What is the formula for \( h \) for such a curve? It is our purpose in this paper to derive this formula.

As a first step in the derivation, let \( q = 2 \). Then \( C^N \) or \( C^{m_{nm}} \) is the intersection of two varieties \( V_{r_1}^n, V_{r_2}^n \), where \( r_1 + r_2 = r + 1 \). Let \( h \), be the number of apparent double points on the curve \( C^{m_i} \) in which an \( S_r \) meets \( V_{r_i}^n \). Decompose one of the given varieties, say \( V_{r_1}^n \), into \( n_1 \) \( r_1 \)-spaces having severally \( \frac{1}{2}n_1(n_1-1) - h_1 \) \( (r_1-1) \)-spaces in common. The curve \( C^{m_1} \) in which an \( S_r \) meets the decomposed \( V_{r_1}^n \) is, then, composed of \( n_1 \) lines forming a skew \( n_1 \)-sided polygon with \( \frac{1}{2}n_1(n_1-1) - h_1 \) vertices. Now the curve \( C^{m_{nm}} \) in which \( V_{r_1}^n \) meets the decomposed \( V_{r_1}^n \) is composed of \( n_1 \)

curves all of order $n_2$. If any two of these $n_1$ curves intersect, they must intersect in $n_2$ points lying in one of the $\frac{n_1(n_1-1)}{2} - h_1$ \((r_1-1)\)-spaces mentioned above. Each of these \((r_1-1)\)-spaces contains $n_2$ such points. Hence, the total number of points in which the $n_1$ curves actually intersect severally is seen to be $n_2\left[n_1(n_1-1)/2 - h_1\right]$. The total number of intersections, both actual and apparent, of the $n_1$ curves two by two is $n_2^2$. Now each of the $n_1$ curves has $h_2$ apparent double points. Therefore, we conclude that the number $h$ of apparent double points on the curve $C^{n_1n_2}$, proper or improper, is equal to the sum of the number of apparent intersections of the component curves of the degenerate $C^{n_1n_3}$ and the total number of the apparent double points on the component curves, that is,

\[
(3) \quad h = \frac{n_1n_2^2}{2}(n_1 - 1) - n_2\left[n_1(n_1-1)/2 - h_1\right] + n_1h_2 \\
= \frac{n_1n_2(n_1-1)}{2} - n_1 - n_2 + 1 + n_2h_1 + n_1h_2.
\]

Suppose we have a curve $C^{n_1n_2n_3}$ which is the intersection of three varieties $V_{n_1}^{n_2}$, $V_{n_2}^{n_3}$, $V_{n_3}^{n_4}$ in $S_{r_1}$, where $r_1 + r_2 + r_3 = 2r + 1$. Let $h_i$ be the number of apparent double points on the curve $C^{n_i}$ in which an $S_{r-1}$ meets $V_{n_i}^{n_i}$. To find the number $h$ of apparent double points on $C^{n_1n_2n_3}$, we may reason as above or we may proceed as follows.

The curve $C^{n_1n_2n_3}$ may be considered as the intersection of $V_{n_1}^{n_2}$ and the variety $V_{n_2}^{n_3} + V_{n_2}^{n_3}$, the latter being the intersection of $V_{n_2}^{n_2}$ and $V_{n_3}^{n_3}$. Let $h_{12}$ be the number of apparent double points on the curve $C^{n_1n_2n_3}$ in which an $S_{r-1}$ meets $V_{n_1}^{n_2}$. Applying formula (3), we find, replacing $n_1$, $n_2$, $h_1$, $h_2$ by $n_1n_2$, $n_3$, $h_{12}$, $h_3$ respectively,

\[
(4) \quad h = \frac{n_1n_2n_3(n_1n_2n_3 - n_1n_2 - n_3 + 1)}{2} + n_2h_{12} + n_1n_2h_3.
\]

Writing for $h_{12}$ its value from (3) in the above, we obtain

\[
(4) \quad h = \frac{n_1n_2n_3(n_1n_2n_3 - n_1 - n_2 - n_3 + 2)}{2} \\
+ n_2n_3h_1 + n_3n_1h_2 + n_1n_3h_2
\]

as the number of apparent double points on $C^{n_1n_2n_3}$.

Now let $q = 4$. Then $C^N$, where $N = n_1n_2n_3n_4$, is the intersection of four varieties $V_{n_1}^{n_1}$, $V_{n_2}^{n_2}$, $V_{n_3}^{n_3}$, $V_{n_4}^{n_4}$, where $r_1 + r_2 + r_3 + r_4 = 2r_1 + 1$. We may regard $C^N$ as the intersection of $V_{n_1}^{n_1}$ and the variety $V_{n_2}^{n_2n_3n_4}$, the latter being the intersection of $V_{n_2}^{n_1}$, $V_{n_3}^{n_1}$, $V_{n_4}^{n_1}$, and
apply (3) and (4), or we may regard it as the intersection of a $V^{n_1 n_2}_{r_1 + r_2 - r}$ and a $V^{n_3 n_4}_{r_3 + r_4 - r}$, the former being the intersection of $V^n_{r_1}$, $V^n_{r_2}$ and the latter that of $V^n_{r_3}$, $V^n_{r_4}$, and then apply (3) alone. Adopting the latter view, we have, replacing $n_1$, $n_2$, $h_1$, $h_2$ by $n_1 n_2$, $n_3 n_4$, $h_{12}$, $h_{34}$ respectively in (3),

$$h = \frac{1}{2} n_1 n_2 n_3 n_4 (n_1 n_2 n_3 n_4 - n_1 n_2 - n_3 n_4 + 1) + n_1 n_2 h_{34} + n_3 n_4 h_{12},$$

where $h_{12}$ and $h_{34}$ are the respective numbers of apparent double points on the curves $C^{n_1 n_2}$, $C^{n_3 n_4}$ in which an $S_{r_1 + r_2 - r}$ and an $S_{r_3 + r_4 - r}$ meet the varieties $V^{n_1 n_2}_{r_1 + r_2 - r}$ and $V^{n_3 n_4}_{r_3 + r_4 - r}$ respectively. Now $h_{12}$ is given by (3) and $h_{34}$ is also given by (3) if $n_1$, $n_2$, $h_1$, $h_2$ are replaced by $n_3$, $n_4$, $h_3$, $h_4$. Making these substitutions in the above, we have

$$h = \frac{1}{2} n_1 n_2 n_3 n_4 (n_1 n_2 n_3 n_4 - n_1 - n_2 - n_3 - n_4 + 3) + n_2 n_3 n_4 h_1 + n_3 n_4 n_1 h_2 + n_4 n_1 n_2 h_3 + n_1 n_2 n_3 h_4.$$

Without going through any further details we give at once the following formula, which can be easily verified, for the number of apparent double points on a curve $C^N$, where $N = n_1 n_2 \cdots n_q$:

$$h = \frac{1}{2} n_1 n_2 \cdots n_q (n_1 n_2 \cdots n_q - \sum n_i + q - 1) + n_1 n_2 \cdots n_q \sum h_i / n_i.$$

If $q = r - 1$, we have, from (2), $r_1 = r_2 = \cdots = r_{r-1} = r - 1$. Then the curve $C^N$ is the intersection of $r - 1$ hypersurfaces. In this case, $h_1 = h_2 = \cdots = h_{r-1} = 0$ as a plane section of a hypersurface cannot have apparent double points. Then (6) is reduced to (1).

As an illustration, let $C^9$ be the intersection of a $V^3_8$ and a $V^3_8$ in $S_5$. Since an $S_3$ in $S_5$ meets $V^3_8$ and $V^3_8$ each in a twisted cubic curve, we have $h_1 = h_2 = 1$. We may use (3) or we may use (6) for $q = 2$. Putting $n_1 = n_2 = 3$, we have $h = 24$ as the number of apparent double points on the curve $C^9$. 

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