1. Introduction. The usual method of determining the Plücker numbers for plane curves is generally laborious. However, in certain cases, by the use of special theorems, the numbers may be determined at once. Thus, the curve \( x_1 : x_2 : x_3 = f(t) : \phi(t) : \psi(t) \), where \( f, \phi, \psi \) are polynomials of degree 3 in \( t \) and the coefficients of \( t^2 \) in each are zero, has one cusp.† Also any cusp or node at a vertex of the triangle of reference is easily recognized by the form of the equation. The purpose of this paper is to derive some of the properties of two functions that are useful in determining whether the rational cubic is nodal or cuspidal.

2. The Cubic Circumscribed about the Triangle of Reference. The parametric equations of the cubic are

\[
(p_i) x_i = (\lambda_i - \lambda_i)(\lambda_i - \lambda_{i+2})(\lambda_i - s_i), \quad (i = 1, 2, 3),
\]

where \( \lambda_i, \lambda_{i+2}, s_i \) are the points of intersection of the cubic with the side \( x_i \) of the triangle, \( \lambda_i \) and \( \lambda_{i+2} \) being vertices. Let \( A \) represent the function

\[
(\lambda_3 - s_1)(\lambda_1 - s_2)(\lambda_2 - s_3) - (\lambda_1 - s_1)(\lambda_2 - s_2)(\lambda_3 - s_3).
\]

**Theorem 1.** The cubic (1) has a cusp at one of the vertices of the triangle of reference or is nodal when \( A \) vanishes.

**Proof.** The class of the cubic is given by the degree of \( \lambda \) in the equation of a tangent line to the curve from any point \((x_1, x_2, x_3)\), after all common factors are eliminated. The tangent line is given by the equation

\[
\sum_{i=1}^{3} x_i (a_i \lambda^4 + b_i \lambda^3 + c_i \lambda^2 + d_i \lambda + e_i) = 0,
\]

where

\[
a_i = -\lambda_i + \lambda_{i+2} + s_{i+2} - s_{i+1},
\]

\[
e_i = \lambda^2_{i+1}(\lambda_i \lambda_{i+2}s_{i+2} + \lambda_{i+2}s_{i+1}s_{i+2} - \lambda_i s_{i+1}s_{i+2} - \lambda_{i+2}s_{i+1}),
\]

* Presented to the Society, June 13, 1931.
† For the general theorem, see Hilton, *Plane Algebraic Curves*, 1920, p. 151.
and $b_i, c_i, d_i$ are other functions of the same quantities. It is readily seen that not more than one of $a_1, a_2, a_3$ can be zero without all being zero. When $A = a_1 = 0$,

$$a_2 = 2 \frac{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)}{(s_3 - 2\lambda_2 + \lambda_3)} \neq 0,$$

and

$$a_3 = 2 \frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{(s_2 - 2\lambda_2 + \lambda_1)} \neq 0.$$

Also if $a_2 = 0$, then $a_1$ and $a_3$ are not zero; or if $a_3 = 0$, $a_1$ and $a_2$ are different from zero. Hence the cubic is nodal unless the coefficients of $x_1, x_2, x_3$ in (2) have a common factor.

Suppose the coefficients of $x_1, x_2, x_3$ have a common factor $\lambda + \alpha$. Make the transformation $\lambda = \lambda' - \alpha$ on $\lambda$. The equation of the tangent line to the curve from any point $(x_1, x_2, x_3)$ is given by (2) with primes on all the letters in the coefficients of $x_1, x_2, x_3$. In particular,

$$e_i = \lambda_i' (\lambda_i' + s_i' s_{i+2}' + \lambda_i' s_{i+1}' s_{i+2}' - \lambda_i' s_i' s_{i+1}' - s_i' s_{i+1}' s_{i+2}' - \lambda_i' \lambda_i' s_i' s_{i+1}' s_{i+2}' ),
\quad (i = 1, 2, 3),$$

where $\lambda_i' = \lambda_i + \alpha, s_i' = s_i + \alpha$. Under this transformation it is seen that

$$A' = (\lambda_3' - s_3' - s_4')(\lambda_4' - s_4') - (\lambda_4' - s_4')(\lambda_3' - s_3') (\lambda_3' - s_4') = A.$$

As $\lambda + \alpha$ is a factor of the old coefficients, $\lambda'$ is a factor of the new ones. Therefore $e_1', e_2', e_3'$ are all zero. This happens when and only when $\lambda_i' = s_i' = s_{i+1}' = 0$, which gives a cusp at a vertex.

It should be noted in the application of this theorem that a cubic with a cusp at a vertex of the triangle of reference is easily recognized, for a common square factor appears in two of the three parametric equations of the curve. Then, when $A$ vanishes, the cubic is nodal if it is not of the above type.

**Theorem 2.** If the cubic (1) is cuspidal, $A$ is zero for those cubics for which the cusp falls at one of the vertices of the triangle of reference, and different from zero for all others.
PROOF. Either \( a_1, a_2, a_3 \) vanish or the coefficients of \( x_1, x_2, x_3 \) have a common factor for a cuspidal cubic. If \( a_1 = a_2 = a_3 = 0 \), \( A = 2(s_1 - s_2) (s_3 - s_2) (s_3 - s_1) \neq 0 \). If the coefficients of \( x_1, x_2, x_3 \) have a common factor, \( \lambda + \alpha \), make the transformation \( \lambda = \lambda' - \alpha \).

The resulting constant terms, \( e_1', e_2', e_3' \), are all zero. It is easily shown that \( A' = A \neq 0 \) unless \( \lambda_1' = s_1' = s_2' = 0, \lambda_2' = s_2' = s_3' = 0 \) or \( \lambda_3' = s_1' = s_3' = 0 \), in which cases a cusp falls at a vertex of the triangle of reference and \( A = 0 \). For these cases \( \lambda_1 = s_1 = s_2 = \alpha, \lambda_2 = s_2 = s_3 = \alpha, \) or \( \lambda_3 = s_1 = s_3 = \alpha. \)

**Theorem 3.** A triangle of reference, inscribed to any nodal cubic, can always be chosen such that \( A \) vanishes.

**Proof.** By a suitable choice of coordinates any crunodal cubic can be put in the form* \( x_1 : x_2 : x_3 = \lambda_1 (\lambda_2 - 1) : (\lambda_2 - 1) : \lambda_3 \) and any acnodal cubic in the form \( x_1 : x_2 : x_3 = \lambda_1 (\lambda_2 + 1) : (\lambda_2 + 1) : \lambda_3 \). A transformation

\[
\begin{align*}
\rho y_1 &= (39x_1 - 14x_2 - 24x_3)/15, \\
\rho y_2 &= (-17x_1 - 30x_2 + 24x_3)/7, \\
\rho y_3 &= (34x_1 - 21x_2 - 24x_3)/10
\end{align*}
\]

puts the crunodal cubic in the form (1), where \( \lambda_1, \lambda_2, \lambda_3, s_1, s_2, s_3 \) are, respectively, 2, 3, \(-7/5\), 1/3, \(-5/7\), 1/2 and for which \( A = 0 \). Likewise a transformation can be found for which the acnodal cubic goes into the form (1) with \( A \) vanishing.

3. **The Cubic Inscribed to the Triangle of Reference.** The parametric equations of the cubic are

\[
(3) \quad \rho x_i = (t_i - t)(t_i - r), \quad (i = 1, 2, 3),
\]

where \( t_i \) and \( r_i \) are the parameters of the points of contact and intersection, respectively, of the cubic with \( x_i = 0 \). Let the function

\[
(t_3 - r_1)(t_1 - r_2)(t_2 - r_3) - (t_2 - r_1)(t_3 - r_2)(t_1 - r_3)
\]

be represented by \( B \).

**Theorem 4.** The cubic given by equations (3) has a cusp on one of the lines \( x_i = 0, (i = 1, 2, 3) \), or is nodal when \( B \) vanishes.

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The proofs for this theorem and the next two are very similar to the proofs of the first three and hence will not be given. The cases where a cusp falls on the lines \(x_i = 0\) may easily be recognized. Thus if any two of \(t_1, t_2, t_3\) are equal, a cusp falls at a vertex of the triangle of reference, and if the corresponding pair from \(r_1, r_2, r_3\) are equal, \(B\) vanishes. A cusp on the lines \(x_i = 0\) but not at a vertex may be found by noting that \(t_1, t_2, t_3\) are the only possible parameters for such a cusp and that the tangent line through it is indeterminate. In this case also \(B\) may vanish.

**Theorem 5.** If the cubic (3) is cuspidal, \(B\) may or may not be zero when the cusp is on one of the lines \(x_i = 0, (i = 1, 2, 3)\), and \(B\) is different from zero for all other cases.

**Theorem 6.** A triangle of reference, circumscribed to any nodal cubic, can always be chosen such that \(B\) vanishes.

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**AN APPLICATION OF METRIC GEOMETRY TO DETERMINANTS**

**BY L. M. BLUMENTHAL**

1. **Introduction.** A paper presented to the Accademia dei Lincei by B. Segre† is devoted to the following theorem, announced by H. W. Richmond:‡

   If in a non-vanishing, symmetric determinant of order six, the six elements in the principal diagonal are all zero, and the complementary minors of five of these elements are also zero, then the complementary minor of the remaining element must be zero.

   Segre states that the analogous theorem for determinants of the second§ and the fourth orders may be immediately verified, and the object of his investigation is to ascertain if analogous theorems are valid for determinants of other orders. He shows

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* Presented to the Society, September 9, 1931.
† *Intorno ad una proprietà dei determinanti simmetrici del \(6^o\) ordine*, Atti dei Lincei, (6), vol. 2 (1925), p. 539.
§ The theorem is, of course, trivial for second-order determinants.