The purpose of this note is to give conditions under which the Stieltjes integral equation

\[ \phi(x) = f(x) + \lambda \int_a^b K(x, y) d\phi(y) \]

has a solution \( \phi(x) \) of bounded variation.

In the first theorem conditions on \( f(x) \) and \( K(x, y) \) are given under which the method of successive substitutions yields a solution of bounded variation for a limited range of values of \( \lambda \).

With further restrictions on \( K(x, y) \), it is shown in the second theorem that the Fredholm method applies and the solution of bounded variation thus obtained is valid for all values of \( \lambda \) except for the characteristic values.

Finally, an example is given to show that the more restrictive conditions on \( K(x, y) \) given in the second theorem are not sufficient to make the problem a special case of that treated by Riesz.†

**Theorem 1.** If

(a) \( f(x) \) is of bounded variation, \( a \leq x \leq b \),
(b) \( K(x, y) \), defined and bounded on \( R(a \leq x \leq b, a \leq y \leq b) \), is continuous in \( y \) for each \( x \) and has a total variation in \( x \) for each \( y \), \( T_K(y) \), which is a bounded function of \( y \) having the least upper bound \( T_K \) and
(c) \[ |\lambda| < 1/T_K, \]

then the function \( \tilde{\phi}(x) \) defined by the series

\[ \tilde{\phi}(x) = f(x) + \lambda \int_a^b K(x, y_1) d\phi(y_1) \]
\[ + \lambda^2 \int_a^b K(x, y_1) d \int_a^b K(y_1, y_2) d\phi(y_2) + \cdots \]

is the unique solution of bounded variation of integral equation (1).

* Presented to the Society, December 31, 1930.
This theorem is proved by the usual method of successive substitutions. Let the function \( \psi(x) \) be defined by
\[
\psi(x) = \int_a^b K(x, y) df(y).
\]

Designate the total variations of \( f(x) \) and \( \psi(x) \) by \( Tf \) and \( T\psi \) respectively and let \( M \geq |K(x, y)| \). Then we have
\[
|\psi(x)| \leq MTf \quad \text{and} \quad T\psi \leq TK \cdot Tf.
\]

With the use of these inequalities it is found that series (2) obtained from equation (1) by repeated substitution converges absolutely and uniformly in \( x \) for all values of \( \lambda \) satisfying the inequality \( |\lambda| < 1/TK \). The function thus defined, \( \bar{\psi}(x) \), is found by substitution to satisfy equation (1). That it is the unique solution follows as a consequence of the fact that the method of successive substitutions when applied to the homogeneous equation
\[
\phi(x) = \lambda \int_a^b K(x, y) d\phi(y)
\]
yields as its only solution \( \phi(x) \equiv 0 \).

**Theorem 2.** If \( f(x) \) is of bounded variation in the interval \((a, b)\) and \( K(x, y) \) together with \( \partial K(x, y)/\partial x \) are continuous functions of \( x \) and \( y \) in \( R \), then there exists a unique solution \( \phi(x) \) of bounded variation of equation (1) for all values of \( \lambda \) except for the characteristic values.

This theorem can be proved by applying Fredholm's method to equation (1). However, we shall employ the following transformation* which reduces the problem to the solution of a Riemann integral equation. Let \( \theta(x) = \phi(x) - f(x) \). Then equation (1) becomes
\[
\theta(x) = \lambda \int_a^b K(x, y) df(y) + \lambda \int_a^b K(x, y) d\theta(y).
\]

It is easily verified, with the given hypotheses on the functions involved, that each term of the right hand side of equation (3) is a continuous function of \( x \) and possesses a continuous

* J. D. Tamarkin suggested to me the possibility of transforming equation (1) into a Riemann integral equation.
derivative. Hence on placing \( \lambda \int_a^b K(x, y)df(y) = F(y) \), we obtain from equation (3) by differentiation

\[
\theta'(x) = F'(x) + \lambda \int_a^b \frac{\partial}{\partial x}K(x, y)\theta'(y)dy.
\]

The function \( F'(x) \) is continuous, and, moreover, by hypothesis \( \partial K(x, y)/\partial x \) is continuous in \( x \) and \( y \). Hence the Riemann integral equation (4) has a unique continuous solution \( \theta'(x) \) for all values of \( \lambda \) except for the characteristic values. The unique solution of bounded variation \( \phi(x) \) of equation (1) can thus be found.

An example. Let \( ||f|| \) denote the maximum of the absolute value of the continuous function \( f(x) \) in \((a, b)\). One of the conditions on the transformation

\[
T[f] = \int_a^b K(x, y)df(y)
\]

as given by F. Riesz* is that there exists a constant \( M \) such that for all continuous functions \( f(x) \)

\[
||T[f]|| \leq M||f||.
\]

The function \( K(x, y) \) defined in the following example satisfies the conditions of Theorem 2 whereas inequality (5) given by Riesz is not satisfied by the transformation \( T[f] \). We define \( K(x, y) \) to be a function of one variable, thus:

\[
K(x, 0) = 0, \quad K(x, y) = y \sin(\pi/y), \quad (0 < y \leq 1).
\]

We next define a sequence of continuous functions \( f_n(y) \) bounded in \( n \) and \( y \). The \( n \)th member of this sequence is a function whose graph consists of a series of broken lines. These lines have a slope equal to zero in the interval \((0, 1/(n+1))\), while in the interval \((1/(n+1), 1)\) they have a negative slope where the function \( y \sin(\pi/y) \) is negative and a positive slope where this function is positive. Let \( \delta_i \) denote 1 or 0 according as \( i \) is odd or even. Then

\[
f_n(y) = (-1)^k(k + 1)y + (-1)^{k+1}k + \delta_k, \quad 1/(k + 1) \leq y \leq 1/k, \quad (k = 1, 2, \ldots, n),
\]

* F. Riesz, loc. cit., p. 72.
and

\[ f_n(y) = \delta_n, \quad (0 \leq y \leq 1/(n + 1)). \]

From this definition we have

\[
\frac{df_n(y)}{dy} = (-1)^k k(k + 1),
\]

\[ (1/(k + 1) < y < 1/k, \ k = 1, 2, \cdots, n), \]

\[ = 0, \quad (0 < y < 1/(n + 1)). \]

The transformation

\[ T[f_n] = \int_0^1 y \sin(\pi/y) df_n(y), \quad (n = 1, 2, \cdots), \]

becomes from the definition of \( f_n(y) \) and from (6)

\[
T[f_n] = \sum_{k=n}^{k=n} \int_{1/(k+1)}^{1/k} y \sin(\pi/y)(-1)^k k(k + 1) dy
\]

\[ = \sum_{k=n}^{k=n} k(k + 1) \int_{1/(k+1)}^{1/k} (-1)^k y \sin(\pi/y) dy. \]

The \( \int_{1/(k+1)}^{1/k} y \sin(\pi/y) dy \) is in absolute value greater than the area of the triangle whose vertices are at \((1/(k + 1), 0), (1/k, 0), [2/(2k + 1), (-1)^k 2/(2k + 1)]\). Since the area of this triangle is \(1/[k(k + 1)(2k + 1)]\), we have

\[
\int_{1/(k+1)}^{1/k} (-1)^k y \sin(\pi/y) dy > 1/[k(k + 1)(2k + 1)].
\]

Consequently we get from (7)

\[
T[f_n] > \sum_{k=n}^{k=n} k(k + 1)/[k(k + 1)(2k + 1)] = \sum_{k=1}^{k=n} 1/(2k + 1),
\]

from which it follows that \( T[f_n] \) becomes infinite with \( n \). Hence condition (5) is not satisfied and equation (1) is not a special case of that treated by F. Riesz.

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