1. Definitions and Notation. The familiar Cesàro transform of a double series \( \sum_{i,j=1}^{\infty} u_{ij} \) is given by

\[
S_{mn}^{(\alpha, \beta)} = S_{mn}^{(\alpha, \beta)} \left[ \binom{m + \alpha - 1}{\alpha} \binom{n + \beta - 1}{\beta} \right],
\]

where

\[
S_{mn}^{(\alpha, \beta)} = \sum_{i,j=1}^{m,n} \binom{m + \alpha - i}{\alpha} \binom{n + \beta - j}{\beta} u_{ij}.
\]

The series \( \sum u_{ij} \) is said to be summable \((C; \alpha, \beta)\) to \( S \) if we have \( \lim_{m,n \to \infty} S_{mn}^{(\alpha, \beta)} = S \); to be bounded \((C; \alpha, \beta)\) if \( |S_{mn}^{(\alpha, \beta)}| < \text{const.} \) for all values of \( m \) and \( n \); and to be ultimately bounded \((C; \alpha, \beta)\) if \( \lim \sup_{m,n \to \infty} |S_{mn}^{(\alpha, \beta)}| < \infty \). This definition holds for all values of \( \alpha \) and \( \beta \), real or complex, except negative integers, the binomial coefficients being defined as usual in terms of the gamma function; however we shall be concerned only with real orders greater than \(-1\).

A special but important type of double series is that for which \( u_{ij} \) is factorable, say

\[
u_{ij} = v_i w_j, \quad (i, j = 1, 2, 3, \ldots).
\]

Defining

\[
V_m^{(\alpha)} = \sum_{i=1}^{m} \binom{m + \alpha - i}{\alpha} v_i; \quad W_n^{(\beta)} = \sum_{j=1}^{n} \binom{n + \beta - j}{\beta} w_j,
\]

we have, by (2), when (3) holds, \( S_{mn}^{(\alpha, \beta)} = V_m^{(\alpha)} W_n^{(\beta)} \), so that, by (1),

\[
S_{mn}^{(\alpha, \beta)} = \tilde{V}_m^{(\alpha)} \tilde{W}_n^{(\beta)},
\]

where the factors in the right member of (5) are, respectively, the \((C, \alpha)\) transform of \( \sum v_i \) and the \((C, \beta)\) transform of \( \sum w_j \).

2. Examples. The relation (5) enables us to obtain very easily examples illustrating the following statements.
THEOREM 1. There is a series $\sum u_{ij}$ which is summable and bounded $(C; \alpha, \beta)$ for every $\alpha > 0, \beta > 0$, while (a) each row and column of $\sum u_{ij}$ has bounded partial sums and (b) each row and column of $\sum u_{ij}$ is non-summable $(C, \gamma)$ for every $\gamma > -1$.

Let $v_i = -1$ and $v_i = 2(-1)^i$ when $i > 1$; then $\sum_{i=1}^{m} v_i = (-1)^m$ and $\sum v_i$ is, as is well known, summable $(C, \delta)$ to 0 for every $\delta > 0$.* Let $\sum w_j$ be any series whose partial sums are bounded, and which is non-summable $(C, \gamma)$ for every $\gamma > -1$.† It is easy to show that the series whose general term is given by

$$u_{ij} = v_i w_j + w_i v_j$$

is summable $(C; \alpha, \beta)$ to 0, and satisfies the other conditions of Theorem 1.

THEOREM 2. Corresponding to each pair of numbers $\alpha$ and $\beta$, $\alpha > -1, \beta > -1$, there is a series $\sum u_{ij}$ which is summable $(C; \alpha, \beta)$ while (a) each row [column] is unbounded $(C, \beta - \delta) [\langle C, \alpha - \delta \rangle], (b)$ each row [column] is non-summable but bounded $(C, \beta) [\langle C, \alpha \rangle]$ and (c) each row [column] is summable $(C, \beta + \delta) [\langle C, \alpha + \delta \rangle], \forall \delta > 0$.

Let $v_i = -1$ and $v_i = 2(-1)^i$ when $i > 1$, as before. Corresponding to a number $\gamma > -1$, let $\sum w_i^{(\gamma)}$ be the series having

$$\sum_{i=1}^{p} (-1)^i \binom{i + \gamma - 1}{\gamma}, \quad (p = 1, 2, 3, \ldots)$$

for its sequence of partial sums. Then $\sum w_i^{(\gamma)}$ has‡ an unbounded $(C, \gamma - \delta)$ transform, a non-convergent but bounded $(C, \gamma)$ transform, and a convergent $(C, \gamma + \delta)$ transform. These facts and properties of $\sum v_i$ enable us to show that the series whose general term is given by $u_{ij} = v_i w_j^{(\gamma)} + w_i^{(\gamma)} v_j$ is summable $(C; \alpha, \beta)$ to 0 and fulfills the other conditions of Theorem 2.

* $\sum v_i$ is bounded $(C, 0)$ and summable $(C, 1)$ to 0; it is therefore summable $(C, \delta)$ to 0 for every $\delta > 0$. See Zygmund, *Sur un théorème de la théorie de la sommabilité*, Mathematische Zeitschrift, vol. 25 (1926), p. 291.

† An example of such a series is $\sum w_j$, where $w_j$ is the coefficient of $x^j$ in the series $\sum_{n=0}^{\infty} (-1)^n x^n$. See Hardy, *On certain oscillating series*, Quarterly Journal of Mathematics, vol. 38 (1906–7), p. 286. Hardy’s result shows that $\sum w_j$ is not Abel summable, and non-summability $(C, \gamma)$ for every $\gamma > -1$ follows.

‡ See Knopp, *Unendliche Reihen*, 3d edition, 1931, p. 496, Ex. 2. The discussion is given for $\gamma = k$, an integer, but it holds also for any real $\gamma > -1$. 
3. **Necessary Conditions for Summability.** We now give two necessary conditions for summability of double series.

**Theorem 3.** If a double series is ultimately bounded \((C; k, l)\), \(k\) and \(l\) being fixed positive integers, then each sufficiently advanced row [column] is bounded \((C, l)\) \([(C, k)]\).

By the ultimate boundedness \((C; k, l)\) there exist constants \(K, M,\) and \(N\) such that \(|S^{(k, l)}_{mn}| < K\) for \(m > M, n > N\). Fix \(m > M + k + 1\); then for every \(n > N, |S^{(k, 0)}_{m-r,n}| < K\), for \(r = 0, 1, 2, \ldots, k+1\). Moreover, we have

\[
|S^{(k, l)}_{m-r,n}| \left/ \binom{m + k - 1}{k} \binom{n + l - 1}{l} \right. < K,
\]

and

\[
\left| \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} S^{(k, l)}_{m-r,n} \left/ \binom{m + k - 1}{k} \binom{n + l - 1}{l} \right. \right| < K \sum_{r=0}^{k+1} \binom{k+1}{r} = 2^{k+1} K.
\]

The product of \((\binom{m+k-1}{k})\) by the sum on the left is the \((C, l)\) transform of the \(m\)th row of the double series.* This row is a simple series which is ultimately bounded \((C, l)\) and therefore is bounded \((C, l)\) as was to be shown.

**Theorem 4.** If a double series is summable \((C; k, l)\), \(k\) and \(l\) being fixed positive integers, then each sufficiently advanced row [column] is either (a) summable \((C; l+\delta)\) \([(C, k+\delta)]\) for every \(\delta > 0\) or (b) non-summable \((C, \gamma)\) for every \(\gamma > -1\).

This follows from Theorem 3 and the result of Zygmund, loc. cit.

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* This is seen by equating coefficients of \(x^m y^n\) in the identity

\[
(1 - x)^{k+1} \sum_{i,j=1}^{\infty} S^{(k, l)}_{ij} x^i y^j = (1 - y)^{-(l+1)} \sum_{i,j=1}^{\infty} \sum_{\gamma=1}^{\infty} [H_{ij}] x^i y^j.
\]