ON n-WEBS OF CURVES IN A PLANE

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This note contains a proof of Theorem 4 of the list given by W. Blaschke* in a preceding paper.

If \( t_i = \text{const.} \) represents \( n \) sheaves of curves in a plane, then the maximal number of linearly independent relations

\[
\sum_i U_{ik}(t_i) = 0, \quad (k = 1, \ldots, m, i = 1, \ldots, n),
\]

is

\[
N = \frac{1}{2}(n - 1)(n - 2).
\]

Let (1) be any set of such relations; then we consider \( U_{ik}(t_i) \), \((k = 1, \ldots, m)\), for a fixed \( i \) to be the \( m \) coordinates of a point describing a curve \( p_i(t_i) \) in an affine \( m \)-space.

If we can prove that the curves \( p_i(t_i) \) all lie in parallel linear subspaces of dimension \( N \), our theorem is proved, for this means that between the coordinates of every \( p_i \) there exist linear relations with the same constant coefficients, which express \( m - N \) of the coordinates in terms of the other \( N \). And this means that of the \( m \) relations (1) there can be only \( N \) linearly independent.

If we assume our functions \( U_{ik} \) to be differentiable a suitable number of times, however, this last statement comes down to proving that among the vectors

\[
\frac{d}{dt_i} p_i(t_i) = p_i'(t_i), \quad p_i''(t_i), \quad p_i'''(t_i), \ldots,
\]

there cannot be more than \( N \) linearly independent ones.†

We will prove this for \( n = 5, N = 6 \); the proof can easily be extended to all values of \( n \). To avoid the use of many indices, we will write (1) in the form

\[
p_1(u) + p_2(v) + p_3(r) + p_4(s) + p_5(t) = 0.
\]


† This does not really make it necessary to assume the functions (1) to be analytic; from a certain order \( m \) we can always replace (3) by an existence statement for solutions of a differential equation.
As our parameters $t_i$ are given functions of the coordinates in the plane of our curves, and are all independent functions, we can express them as functions of $u$ and $v$. We then differentiate the vector equation (3) with respect to $u$ and $v$ and find

$$0 = \rho' + \rho_r u + \rho_s v + \rho_t v,$$

$$0 = \rho' + \rho_r u + \rho_s v + \rho_t v;$$

$$0 = \rho''' + \rho'' s + \rho'' t + \rho_r u + \rho_s v + \rho_t v + \rho'' t_{uv},$$

$$0 = \rho'' + \rho'' s + \rho'' t + \rho_r u + \rho_s v + \rho_t v + \rho'' t_{uv};$$

Here $L$ always means a linear combination of the vectors in brackets, and $i = 3, 4, 5$. In this way we get two groups of equations. The first expresses all the derivatives of $\rho_1$ and $\rho_2$ as combinations of those of $\rho_3$, $\rho_4$, and $\rho_5$. The latter can be used to prove that of these there can be no more than 6 linearly independent.

If we assume for a moment that the equations (8), (9), (10) are not in a disturbing way dependent, then the result is obvious. For then we can have at the utmost 3 independent vectors $\rho_1'$, of the vectors $\rho_1''$ one can be expressed by means of the others and $\rho_1'$ as a consequence of (8), so we get only two extra independent vectors, and (9) shows that vectors $\rho_1'''$ can give only one extra dimension. The total number is exactly $3 + 2 + 1 = 6$. 
So the only thing that remains to be proved is that relations (8), (9), (10) are really independent. Now in (8) the coefficients of \( p_i^{r'} \) cannot vanish. For \( r_u = 0 \) would mean that \( r \) was a function of \( u \) alone, and therefore that sheaves \( r = \text{const.} \) and \( u = \text{const.} \) would coincide. So (8) really gives us a relation between the \( p_i^{r'} \). To show that (9) gives 2 relations we have to consider the matrix

\[
\begin{vmatrix}
  r_u^2 r_v & s_u^2 s_v & t_u^2 t_v \\
  r_u r_v^2 & s_u s_v^2 & t_u t_v^2 \\
\end{vmatrix}
\]

and show that it is of rank 2. But one of the determinants is

\[
r_u r_v s_u s_v^2 \\
\]

and none of the factors can vanish, the last one since this would imply the dependence of the functions \( r \) and \( s \), which is again impossible. Finally the essential determinant in (10) is equal to

\[
r_u r_v s_u s_v^2 t_u t_v^2 \\
\]

so that from (10) we can really compute \( p_i^{lv} \) as linear combinations of \( p_i^{r''}, p_i^{r'}, p_i^{r} \). We see that there is no danger for dependency of the equations, and our theorem is proved.

Of course if \( n > 5 \), we have a similar proof, only the determinants we have to consider are of higher order. We find

\[
N = n - 2 + n - 3 + n - 4 + \cdots + 2 + 1 = \frac{1}{2} (n - 1)(n - 2).
\]

As a corollary, for \( n = 4 \), we have: If a 2-dimensional surface in \( k \)-space can be generated in two different ways as a translation surface, it lies in a linear three-dimensional subspace.*

For the assumption leads to a vector equation (4) with \( n = 4 \) and our formula gives \( N = 3 \).

* See S. Lie, Leipziger Berichte, 1897, p. 186.