Since $a \leq 374930473917097$, we have in each case $k \leq 39111579$. Thus the problem of representing $N$ as the difference of squares was split into 8 parts. The first two parts were covered by the machine without any result. On the third run, however, the machine stopped almost at once at $x = 58088$. This gives

$$a = 556846584735, \quad b = 556644555032.$$ 

Hence we have the factorization

$$2^{79} - 1 = 2687 \cdot 202029703 \cdot 1113491139767.$$ 

It is not difficult to show that the factors are primes. This is the 13th composite Mersenne number to be completely factored. The author’s recent report* on Mersenne numbers should be changed accordingly.

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MATRICES Whose $s$TH COMPOUNDS ARE EQUAL

BY JOHN WILLIAMSON

If $A$ is a matrix of $m$ rows and $n$ columns and $s$ is any positive integer less than or equal to the smaller of $n$ and $m$, from $A$ can be formed a new matrix $A_s$ of $mC_s$ rows and $nC_s$ columns, the elements in the $t$th row of $A_s$ being the $nC_s$ determinants of order $s$ that can be formed from the $t_1$th, $\cdots$, $t_s$th rows of $A$, and the elements in the $t$th column being the $mC_s$ determinants of order $s$ that can be formed from the $t_1$th, $\cdots$, $t_s$th columns of $A$. The matrix $A_s$, so defined, is called the $s$th compound matrix of $A$. In the following note we discuss the necessary and sufficient conditions under which the $s$th compounds of two matrices are equal. We shall require the following lemmas.

**Lemma I.** The rank of the $s$th compound of a matrix $A$, whose rank is $r$, is $\leq C_s$ if $r \geq s$ and is zero if $s > r$.†

* This Bulletin, vol. 38 (1932), p. 384. Dr. N. G. W. H. Beeger has kindly called my attention to the fact that $2^{233} - 1$ has two known prime factors and should be classified accordingly.

† Cullis, Matrices and Determinoids, vol. 1, p. 289.
Lemma II. The $s$th compound of the product of two matrices is the product of the $s$th compounds of the two matrices, or, in symbols,*

(1) \[(AB)_s = A_s B_s.\]

Theorem. If $A$ is a matrix of rank $r$, the necessary and sufficient condition that $A_s = B_s$ is that

(a) the rank of $B$ be less than $s$ when $r < s$;
(b) there exist two non-singular matrices $C$ and $D$ such that

\[
CAD = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad CBD = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},
\]

where $T$ and $S$ are two non-singular matrices of $r$ rows and columns such that $|T| = |S|$, when $r = s$;
(c) $A = \omega B$, where $\omega$ is an $s$th root of unity, when $r > s$.

In case (a) if $A_s = B_s$, then $B_s = 0$ and by Lemma I the rank of $B$ is less than $r$. On the other hand if the rank of $B$ is less than $s$, then $B_s = 0 = A_s$. In case (b) the sufficiency of the condition follows from (1) and the fact that

\[
\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}_s = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}_s.
\]

We now proceed to prove that the condition stated above is necessary. Since $A$ has rank $r$ there exist two non-singular matrices $C$ and $D$ such that

\[
CAD = R = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix},
\]

where $T$ is any non-singular $r$-rowed square matrix. If

\[
CBD = F = \begin{pmatrix} S & G \\ H & K \end{pmatrix},
\]

where $S$ is an $r$-rowed square matrix, $G$ an $r$ by $n-r$ matrix, $H$ an $m-r$ by $r$ matrix, and $K$ an $m-r$ by $n-r$ matrix, then, since $A_s = B_s$, it follows that $R_s = F_s$ and $|S| = |T| \neq 0$. Since $R_s$ contains only one element different from zero, every determinant of order $s$ that can be formed from $s-1$ columns of $S$ and one of $G$ is zero. If

$S = (s_{ij}), \quad G = (g_{iq}), \quad (i, j = 1, 2, \ldots r; \quad q = 1, 2, \ldots, n-r),$

and $S_{ij}$ is the cofactor of $s_{ij}$ in $S$, then

\begin{equation}
\sum_{i=1}^{r} S_{ij} g_{iq} = 0, \quad (j = 1, 2, \ldots, r; q = 1, 2, \ldots, n - r).
\end{equation}

For a fixed $q$, the equation (2) represents a set of $r$ homogeneous equations in the $r$ unknowns $g_{iq}$, and since $|S_{ij}| = |S|^{r-1} \neq 0$, it follows that $g_{iq} = 0$. Accordingly $G = 0$ and by a similar argument $H = 0$, so that $F$ has the form

\[
\begin{pmatrix}
S & 0 \\
0 & K
\end{pmatrix}.
\]

But, since $S$ is non-singular, at least one of the quantities $S_{ij} \neq 0$. If $k$ is any element of $K$, we observe that $kS_{ij}$ is an element of $F$, which must be zero, and therefore $K = 0$.

In case (c), the sufficiency of the condition is an immediate consequence of (1). If the rank $r$ of $A$ is greater than $s$, there must exist in $A$ a submatrix $T$ of $s+1$ rows and columns, which is non-singular. Without any loss of generality we may suppose that

\[
A = \begin{pmatrix}
T & K \\
L & M
\end{pmatrix}, \quad B = \begin{pmatrix}
S & H \\
P & Q
\end{pmatrix},
\]

where $S$ is an $(s+1)$-rowed square matrix. From $A_s = B_s$, we deduce that $T_s = S_s$ and

\[
|T_s| = |T|^s = |S_s| = |S|^s,
\]

so that

\begin{equation}
|S| = \omega |T|,
\end{equation}

where $\omega$ is an $s$th root of unity. Moreover *

\[
(T_s)_s = |T|^{s-1} T = (S_s)_s = |S|^{s-1} S,
\]

so that, by (3), $S = \omega T$. Since $T$ is non-singular, there must exist in $T$ a non-singular submatrix $T'$ of $s$ rows and columns. If $A'$ denote a matrix obtained from $A$ by a rearrangement of rows and columns, so that $T'$ occurs in the top left-hand corner of $A'$, and $B'$ is the matrix obtained from $B$ by exactly the same rearrangement, then

* $(T_s)_s$ denotes the $s$th compound of $T$. That $(T_s)_s = |T|^{s-1} T$ is simply the well known theorem on the adjugate of the adjugate of a matrix.
REMARKS ON PRINCIPIA MATHEMATICA

A' = \begin{pmatrix} T' & K' \\ L' & M' \end{pmatrix}, \quad B' = \begin{pmatrix} \omega T' & H' \\ P' & Q' \end{pmatrix},

and from \( A_s = B_s \) it follows that \( A_s' = B_s' \). If

\[
T' = (t_{ij}), \quad K' = (k_{iq}), \quad H' = (h_{iq}),
\]

\((i, j = 1, 2, \ldots, s; q = 1, 2, \ldots, n - s)\),

and \( T_{ij} \) denote the cofactor of \( t_{ij} \) in \( T' \), then

\[
\sum_{i=1}^{s} T_{ij} k_{iq} = \sum_{i=1}^{s} \omega^{s-1} T_{ij} h_{iq}, \quad \text{or} \quad \sum_{i=1}^{s} T_{ij} (k_{iq} - \omega^{s-1} h_{iq}) = 0.
\]

But, since \(|T_{ij}| \neq 0\), \( k_{iq} - \omega^{s-1} h_{iq} = 0 \) or \( H' = \omega K' \). Similarly it may be shown that \( P' = \omega L' \). Let \( T'' \) be a submatrix of \( T' \) of order \( s - 1 \) which is non-singular. If \( m_{ij} \) is any element of \( M' \) and \( q_{ij} \) the corresponding element of \( Q' \), the determinant of order \( s \) formed from \( A' \) of the \( s - 1 \) rows and columns of which \( T'' \) is composed and the row and column in which \( m_{ij} \) lies is equal to the corresponding determinant formed from \( B' \). But from the equality of these two determinants it follows that \( m_{ij} |T''| = \omega^{s-1} q_{ij} |T''| \) and therefore, since \(|T''| \neq 0\), it follows that \( Q' = \omega M' \), \( A' = \omega B' \), and \( A = \omega B \). This completes the proof of the theorem.

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REMARKS ON PROPOSITIONS *1·1 AND *3·35
OF PRINCIPIA MATHEMATICA†

BY B. A. BERNSTEIN

1. Object. Among the propositions of the theory of deduction underlying Whitehead and Russell's *Principia Mathematica* are the two following:

*1·1. Anything implied by a true elementary proposition is true.*

*3·35. \( \vdash p \cdot p \supset q \cdot q \).*

The authors interpret *3·35 as "if \( p \) is true, and \( q \) follows from it, then \( q \) is true," and they remark that *3·35 "differs

† Presented to the Society, September 2, 1932.