HAUSDORFF TRANSFORMATIONS FOR DOUBLE SEQUENCES

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1. Introduction. The purpose of this note is to extend to double sequences some of the results of Hausdorff’s notable papers on methods of summability and moment-sequences.†

Let $\lambda = \| \lambda_{pqmn} \|$, $(p, q, m, n = 0, 1, 2, \cdots )$, be a four-dimensional matrix of real or complex numbers. Then the system of equations

$$A_{pq} = \sum_{m,n=0}^{\infty} \lambda_{pqmn}a_{mn},$$

if the series all converge, transforms a double sequence $\{a_{mn}\}$ into a new double sequence $\{A_{pq}\}$. Necessary and sufficient conditions‡ that this transformation be convergence-preserving for bounded sequences are the following:

(A) $\sum_{m,n=0}^{\infty} |\lambda_{pqmn}| \leq M, \quad (p, q = 0, 1, 2, \cdots ),$

(B) $\lim_{p,q \to \infty} \sum_{m,n=0}^{\infty} \lambda_{pqmn} = l, \quad (m, n = 0, 1, 2, \cdots ),$

(C) $\lim_{p,q \to \infty} \lambda_{pqmn} = l_{mn}, \quad (m, n = 0, 1, 2, \cdots ),$

(D) $\lim_{p,q \to \infty} \sum_{m=0}^{\infty} |\lambda_{pqmn} - l_{mn}| = 0, \quad (n = 0, 1, 2, \cdots ),$


A matrix $\lambda$ satisfying these conditions we call, with Hausdorff, a $C$-matrix. Such a matrix defines a transformation (1) which carries a bounded sequence $\{a_{mn}\}$ convergent to $\alpha$ into a bounded sequence $\{A_{pq}\}$ convergent to

$$l\alpha + \sum_{n=0}^{\infty} l_{mn}(a_{mn} - \alpha),$$

this double series being always absolutely convergent. For a $C$-matrix to define a transformation regular for bounded null sequences it is necessary and sufficient that all the $l_{mn}$ vanish; in such a case the $C$-matrix will be called pure. For a pure $C$-matrix to define a transformation regular for all bounded sequences it is necessary and sufficient that $l = 1$; in this event the $C$-matrix will be called normalized.

Now let $\rho$ be a fixed matrix having an inverse and $\mu = \|\mu_{mnkl}\|$, an arbitrary "diagonal" matrix; that is, $\mu_{mnkl} = \delta_{jk}$ for $k \neq m$ or $l \neq n$ or both, so that the only elements $\neq 0$ are $\mu_{mnmn}(m, n = 0, 1, 2, \ldots)$. For simplicity we write

$$\mu_{mnmn} = \mu_{mn}. $$

Henceforth we restrict ourselves to matrices $\lambda$ of the form†

$$\lambda = \rho^{-1} \cdot \mu \cdot \rho,$$

which by means of the fixed $\rho$ are transformable into diagonal matrices. It is seen at once that any two such matrices are permutable; moreover, the proof is immediate that if $\lambda^* = \rho^{-1} \cdot \mu^* \cdot \rho$ is one of the matrices (2) for which the corresponding diagonal matrix $\mu^*$ has elements $\mu^*_{mn}$ no two of which are equal, then all matrices $\lambda$ permutable with $\lambda^*$ are of the form (2).

The system of equations (1) may be written in matrix form as

$$A = \lambda \cdot a,$$

where $a = \|a_{mnkl}\|$ and $A = \|A_{pqmn}\|$, with

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† By a product such as $\lambda \cdot \mu$ we mean the matrix whose element with indices $\rho, q, k, l$ is $\sum_{m,n=0}^{\infty} \lambda_{pnkl} \mu_{mnkl}$. 

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\[
\begin{align*}
  a_{mnkl} &= \begin{cases} 
  a_{mn} & \text{for } k = l = 0, \\
  0 & \text{otherwise},
\end{cases} \\
  A_{pqmn} &= \begin{cases} 
  A_{pq} & \text{for } m = n = 0, \\
  0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

If we set \( b = \rho \cdot a \), \( B = \rho \cdot A \), the matrix equation (3) becomes

\[
\begin{align*}
  (4) \quad B &= \mu \cdot b, \quad \text{or} \quad B_{mn} = \mu_{mn} b_{mn},
\end{align*}
\]
a multiplication. To the matrix \( \lambda \) there corresponds a unique diagonal matrix \( \mu \), or factor sequence \( \{ \mu_{mn} \} \), and conversely. If \( \lambda \) is a \( C \)-matrix, we call \( \{ \mu_{mn} \} \) a \( C \)-sequence; a \( C \)-sequence will be said to be pure or normalized according as \( \lambda \) is pure or normalized.

2. Difference Sequences. From a double sequence \( \{ a_{mn} \} \) one may form the quadruple sequence of double differences of various orders,

\[
(5) \quad \Delta_{ij} a_{mn} = \sum_{h, i=0}^{i, j} (-1)^{k+l} \binom{i}{k} \binom{j}{l} a_{m+k, n+l}.
\]

The recursion formulas

\[
(6) \quad \Delta_{ij} a_{mn} = \Delta_{ij} a_{m+1, n} + \Delta_{i+1,j} a_{mn}
\]
may at once be derived. For brevity we call the double sequence of numbers

\[
\Delta_{ij} a_{00} = b_{ij} = \Delta_{00} b_{ij}
\]
the difference sequence of the \( a_{mn} \), and from (6) readily follows the relation

\[
\Delta_{ij} a_{mn} = \Delta_{mn} b_{ij},
\]
so that the \( a_{mn} = \Delta_{00} a_{mn} \) also constitute the difference sequence of the \( b_{ij} \). The matrix associated with (5) is the matrix of the Euler transformation for double sequences,

\[
E = E' \odot E',
\]
where \( E' \) denotes the Euler matrix for simple sequence transformations and a notation employed in \( A \) is used.† By Theorem

† That is, if each element \( a_{mnkl} \) of a four-dimensional matrix \( a \) is equal to \( a'_{m} a''_{n} \), where these factors are the elements of two-dimensional matrices \( a' \) and \( a'' \), we write \( a = a' \odot a'' \).
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7 of $A$, $E$ is its own inverse. We now choose $E$ to be the fixed matrix $\rho$ introduced above, and hereafter consider only such systems of equations (1) as have their difference sequences $b_{mn}, B_{mn}$ in the multiplicative form (4).

We may express the equations (4) in terms of $a_{mn}, A_{pq}$ and obtain without difficulty from $A = E \cdot B$ the relation

$$A_{pq} = \sum_{i, j=0}^{p, q} \binom{p}{i} \binom{q}{j} \Delta_{p-i, q-j \mu ij} \cdot a_{ij}.$$ 

Setting all the $a_{ij} = 1$, we have

$$\sum_{i, j=0}^{p, q} \binom{p}{i} \binom{q}{j} \Delta_{p-i, q-j \mu ij} = \Delta_{00\mu 00} = \mu_{00}.$$ 

Then the conditions (A)-(E) become

$$\begin{align*}
&\text{(a)} \sum_{i, j=0}^{p, q} \binom{p}{i} \binom{q}{j} |\Delta_{p-i, q-j \mu ij}| = M_{pq} \leq M, \\
&\text{(b)} \lim_{\rho, \delta, \mu \to \infty} \binom{p}{i} \binom{q}{j} \Delta_{p-i, q-j \mu ij} = l_{ij}, \\
&\text{(c)} \lim_{\rho, \delta, \mu \to \infty} \sum_{i=0}^{p} \binom{p}{i} \binom{q}{j} \Delta_{p-i, q-j \mu ij} - l_{ij} = 0, \\
&\text{(d)} \lim_{\rho, \delta, \mu \to \infty} \sum_{j=0}^{q} \binom{p}{i} \binom{q}{j} \Delta_{p-i, q-j \mu ij} - l_{ij} = 0,
\end{align*}$$

the condition (b) corresponding to (B) being automatically satisfied because of (7). These conditions (a)-(e) are thus necessary and sufficient for a $C$-sequence. A $C$-sequence is pure if and only if all the $l_{ij}$ vanish; a pure $C$-sequence is normalized if and only if $\mu_{00} = 1$.

The corresponding conditions, necessary and sufficient for a simple sequence $\{\mu_m\}$ to be a $C$-sequence, are
(21) \[ \sum_{i=0}^{p} \binom{p}{i} \Delta_{p-i,\mu_i} = M_p \preceq M, \quad (M_p \rightarrow M), \]

(22) \[ \lim_{p \rightarrow \infty} \binom{p}{i} \Delta_{p-i,\mu_i} = l_i, \quad (i = 0, 1, 2, \ldots). \]

Of these the first implies the second with \( l_i = 0 \) for \( i > 0 \). It is therefore natural to inquire to what extent condition (21) implies the three remaining conditions (22)-(24). The complete answer to this question will be given in §4, although even in §3 it will become apparent that (21) does not imply (22) and (24).

3. C-Sequences of Product Type. An interesting class of matrices \( \lambda \) of the form (2) is that for which the diagonal matrix corresponding to \( \lambda \) is the product (see the last footnote above) of (simple) diagonal matrices. The importance of this class is sufficiently indicated by the fact that, by Theorem 6 of A, this class is identical with the class of matrices \( \lambda = \lambda' \odot \lambda'' \), in which \( \lambda' \) and \( \lambda'' \) are transformable by \( E' \) into diagonal matrices.

If \( \{\mu_{mn}\} = \{\mu_m, \mu_n''\} \), we clearly have

\[ \Delta_{p-i,q-j,\mu_{ij}} = \Delta_{p-i,\mu_i} \Delta_{q-j,\mu_j''}. \]

The following theorems may now be established easily.

**Theorem 1.** A double sequence \( \{\mu_{mn}\} = \{\mu_m', \mu_n''\} \), where \( \{\mu_m'\} \) and \( \{\mu_n''\} \) are (simple) C-sequences, is a (double) C-sequence if and only if the following condition is satisfied:

\[ l_0' \mid (M'' - l_0'' \mid M' - l_0') = 0. \]

Here the \( M \) and \( l_0 \) of (21) and (22) are primed to agree with the priming of the \( \mu \)'s. Any double sequence thus factorable into simple C-sequences obviously satisfies (21), but it will satisfy both (22) and (24) if and only if (26) is also fulfilled.

**Theorem 2.** A double sequence \( \{\mu_{mn}\} = \{\mu_m', \mu_n''\} \), where \( \{\mu_m'\} \) and \( \{\mu_n''\} \) are (simple) C-sequences, is a pure (double) C-sequence if and only if one of its factors is a pure C-sequence and the other is a pure C-sequence or satisfies the condition \( M = l_0 \).

**Theorem 3.** If \( \{\mu_{mn}\} = \{\mu_m', \mu_n''\} \) is a (double) C-sequence that is not pure, both of its factors are C-sequences, neither factor is pure, and \( l_0' l_0'' \) equals \( l_{00} \).
THEOREM 4. If \( \{ \mu_{mn} \} = \{ \mu_m', \mu_n'' \} \) is a pure (double) C-sequence, not all of whose elements are zero, both of its factors are C-sequences and at least one of them is pure.

4. The Relation Between C-Sequences and Moment-Sequences.

Let \( \chi(u, v) \) be a function which in the square \( U(0 \leq u \leq 1, 0 \leq v \leq 1) \) is of bounded variation in the sense of Hardy-Krause; then the sequence \( \{ \mu_{mn} \} \), where

\[
\mu_{mn} = \int_0^1 \int_0^1 u^m v^n d_u d_v \chi(u, v), \quad (m, n = 0, 1, 2, \ldots),
\]

may be termed a (double) moment-sequence.† According to a theorem of Hildebrandt and Schoenberg,‡ if \( \{ \mu_{mn} \} \) is any double sequence satisfying condition \((\alpha)\) for \( q = p \), there exists a function \( \chi(u, v) \) which generates this sequence, and conversely; hence we have the following theorem.

THEOREM 5. Every C-sequence is a moment-sequence.

Clearly not every moment-sequence is a C-sequence; for otherwise, by Hildebrandt and Schoenberg’s theorem, condition \((\alpha)\) for \( q = p \) would imply both \((\delta)\) and \((\epsilon)\), and we have already seen that this is not so. We therefore seek to determine under what circumstances a moment-sequence is a C-sequence.

It should be observed first that since \( \chi(u, v) \) can always be expressed as the difference between two functions \( \chi_1(u, v) \) and \( \chi_2(u, v) \), each of which is bounded, is non-decreasing in \( x \) alone and in \( y \) alone, and satisfies the condition \( \chi_i(u_1, v_1) - \chi_i(u_1, v_2) - \chi_i(u_2, v_1) + \chi_i(u_2, v_2) \geq 0 \) for \( u_2 > u_1, v_2 > v_1 \); and since the difference (or sum) of two C-sequences is a C-sequence, no loss of generality results from assuming \( \chi(u, v) \) to be of this monotonic character. Then we have, for all \( i, j, m, n, \)

† Although apparently it is more general to assume only that \( \chi(u, v) \) is of bounded variation in the sense of Vitali, this is not actually so; for if \( \chi(u, v) \) is of bounded variation in the Vitali sense, there always exists a related function \( \chi'(u, v) \), where \( \chi'(u, v) = \chi(u, v) - \chi(u, 0) - \chi(0, v) + \chi(0, 0) \), such that the Riemann-Stieltjes integral (9) taken with respect to \( \chi'(u, v) \) has the same value, and \( \chi'(u, v) \) is of bounded variation in the sense of Hardy-Krause. Moreover, \( \chi'(u, v) \) vanishes along \( u = 0 \) and \( v = 0 \); hence it would be no real restriction to assume \( \chi(u, 0) = \chi(0, v) = 0 \). We do not make this assumption, however, for reasons of symmetry.

so that \( \{ \mu_{mn} \} \) is completely monotonic, and by (7), condition (\( \alpha \)) is satisfied for all \( p, q \). We shall now show that the remaining conditions (\( \gamma \))-\((\varepsilon)\) are fulfilled in case \( \chi(u, v) \) satisfies the continuity condition

\[
\chi(u, +0) = \chi(u, 0), \quad \chi(u, +0) = \lim_{v \to 0} \chi(u, v), \quad 0 \leq u \leq 1;
\]

\[
(10) \quad \chi(+0, v) = \chi(0, v), \quad \chi(+0, v) = \lim_{u \to 0} \chi(u, v), \quad 0 \leq v \leq 1.
\]

For all values of \( u \) and \( v \) in \( U \) we have

\[
(11) \quad \left( \begin{array}{c}
p \\ i 
\end{array} \right) u^i (1 - u)^{p - i} \leq \sum_{i=0}^{p} \left( \begin{array}{c}
p \\ i 
\end{array} \right) u^i (1 - u)^{p - i}
\]

\[
= [u + (1 - u)]^p = 1,
\]

together with a like relation in \( v \). Hence, for \( 0 < \delta < 1 \), we obtain

\[
\left( \begin{array}{c}
p \\ i 
\end{array} \right) \left( \begin{array}{c}
q \\ j 
\end{array} \right) \Delta_{p-i, q-j} \mu_{ij} \leq \int_0^\delta \int_0^1 d_u d_v \chi(u, v)
\]

\[
+ \left( \begin{array}{c}
p \\ i 
\end{array} \right) (1 - \delta)^{p-i} \int_\delta^1 \int_0^1 d_u d_v \chi(u, v).
\]

Consequently, for each \( \delta \), we have

\[
\lim_{p, q \to \infty} \left( \begin{array}{c}
p \\ i 
\end{array} \right) \left( \begin{array}{c}
q \\ j 
\end{array} \right) \Delta_{p-i, q-j} \mu_{ij}
\]

\[
\leq \chi(\delta, 1) - \chi(0, 1) - \chi(\delta, 0) + \chi(0, 0),
\]

and so, by virtue of (10), the limit exists and equals zero for each \( i \) and each \( j \). Thus condition (\( \gamma \)) is satisfied with all \( I_{ij} = 0 \). That (\( \delta \)) and (\( \varepsilon \)) are fulfilled then follows in a precisely similar manner, proper account being taken of the relations (11).

When \( \chi(u, v) \) does not satisfy condition (10), let

\[
J_1(u) = \chi(u, +0) - \chi(u, 0), \quad u > 0;
\]

\[
J_2(v) = \chi(+0, v) - \chi(0, v), \quad v > 0;
\]

\[
J_1(0) = J_2(0) = \lim_{0, u \to +0, 0, u \to -0, v \to 0} \chi(u, v) - \chi(0, 0).
\]
Since $\chi(u, v)$ is bounded and monotonic, all the limits involved exist, and $J_1(u)$ and $J_2(v)$ are functions of bounded variation. Moreover, condition (10) is satisfied by the function
\[ \chi^*(u, v) = \chi(u, v) + J(u, v), \]
where
\[ J(u, v) = \begin{cases} 
J_1(u) & \text{for } v = 0, \\
J_2(v) & \text{for } u = 0, \\
0 & \text{for } u, v > 0.
\end{cases} \]

Therefore it remains only for us to investigate the sequence $\{\mu_{mn}\}$ generated by $J(u, v)$.

From the definition of the integral in (9), we have at once
\[ \mu_{00} = J_1(0) - J_1(1) - J_2(1), \]
\[ \mu_{m0} = - \int_0^1 u^m dJ_1(u), \quad (m = 1, 2, 3, \cdots), \]
\[ \mu_{0n} = - \int_0^1 v^n dJ_2(v), \quad (n = 1, 2, 3, \cdots), \]
\[ \mu_{mn} = 0, \quad (m, n = 1, 2, 3, \cdots). \]

Thus $\{\mu_{m0}\}$, $(m = 0, 1, 2, \cdots)$, and $\{\mu_{0n}\}$, $(n = 0, 1, 2, \cdots)$, are both (simple) C-sequences. By (13) we have
\[ M_{pq} = \sum_{i=0}^{p} \binom{p}{i} |\Delta_{p-i,0}\mu_{i0}| + \sum_{j=0}^{q} \binom{q}{j} |\Delta_{0,q-j}\mu_{0j}| - |\Delta_{pq}\mu_{00}|. \]

Both sums are bounded, and the remaining term on the right is also bounded, since
\[ \Delta_{pq}\mu_{00} = \Delta_{p0}\mu_{00} + \Delta_{0q}\mu_{00} - \mu_{00} \]
is the sum of a term depending upon $p$ alone and a term depending upon $q$ alone, each of which tends to a limit as $p$ and $q$ become infinite. Therefore $(\alpha)$ is satisfied.

Condition $(\gamma)$ is obviously satisfied for $i, j > 0$, with $l_{ij} = 0$. For $i = 0, j > 0$, the quantity in question reduces to
\[
\left( \begin{array}{c}
\frac{q}{j}
\end{array} \right) \Delta_{0, q-j\mu_0 j, i,
\]
which tends to zero with \(1/q\); for \(i > 0, j = 0\), the situation is similar. For \(i = j = 0\), the quantity in question reduces to (14), which, as we have already observed, tends to a limit. Therefore (\(\gamma\)) is fulfilled, with \(l_{ij} = 0\) for \(i, j \neq 0, 0\) and

\[l_{00} = \lim_{p \to \infty} \Delta_{0p\mu_{00}} + \lim_{q \to \infty} \Delta_{0q\mu_{00}} = \mu_{00}.\]

For \(j > 0\), the sum in (\(\delta\)) reduces to the absolute value of (15), which tends to zero with \(1/q\). For \(j = 0\), the sum reduces to

\[| \Delta_{pq\mu_{00}} - l_{00} | + \sum_{i = 0}^{p} \left( \begin{array}{c}
\frac{p}{i}
\end{array} \right) | \Delta_{p-i, 0\mu_0 i, 0} | - | \Delta_{p0\mu_0 0} | ,
\]
of which the first term tends to zero but the remaining part in general does not.

For \(i > 0\), the sum in (\(\epsilon\)) reduces to

\[\left( \begin{array}{c}
\frac{p}{i}
\end{array} \right) | \Delta_{p-i, 0\mu_0 i, 0} | ,
\]
which tends to zero with \(1/p\). For \(i = 0\), the sum reduces to

\[| \Delta_{pq\mu_{00}} - l_{00} | + \sum_{j = 0}^{q} \left( \begin{array}{c}
\frac{q}{j}
\end{array} \right) | \Delta_{0, q-j\mu_0 j, 0} | - | \Delta_{0q\mu_0 0} | ,
\]
of which the first term tends to zero but again the remaining part in general does not. The results of this discussion may be summed up as follows.

**Theorem 6.** Let \(J_1(u)\) and \(J_2(v)\) represent the jump of \(\chi(u, v)\) at the sides \(v = 0\) and \(u = 0\), respectively, of the square \(U\) (for orthogonal approach except at \((0, 0)\), where the approach is made in an arbitrary manner from the interior of \(U\)). \(J_1(u)\) and \(J_2(v)\) are functions of bounded variation and so generate (simple) moment-sequences which we may denote by \(\{j_{m(1)}\}\) and \(\{j_{m(2)}\}\), respectively. The sequences \(j_{0(1)} - J_1(1)\), \(j_{1(1)}\), \(j_{2(1)}\), \ldots and \(j_{0(2)} - J_1(1) = j_{0(1)} - J_2(1)\), \(j_{1(2)}\), \(j_{2(2)}\), \ldots, are also moment-sequences and therefore C-sequences; let the \(M\) of \((U)\) and the \(l_0\) of \((C)\) associated with these sequences be denoted respectively by \(M^{(1)}\), \(l_0^{(1)}\) and \(M^{(2)}\), \(l_0^{(2)}\). Then a (double) moment-sequence is a (double) C-sequence if and only if we have
A (double) moment-sequence is a pure (double) C-sequence if and only if in addition to (16) we have

\[ l_0^{(1)} + l_0^{(2)} - f_0^{(1)} + J_2(1) = 0. \]

In particular we note that if \( J_1(u) \equiv J_2(v) \equiv 0, \chi(u, v) \) generates a pure C-sequence.

**Theorem 7.** Condition (α) for \( q = p \) implies the entire set of conditions (α)-(ε), including \( l_{ij} = 0 \) for \( i, j \neq 0, 0 \), with the exception of (δ) for \( j = 0 \) and (ε) for \( i = 0 \).

Although in this section it has been tacitly assumed that the sequences considered are real, the extension of the results to complex sequences is immediate.