NOTE ON A SPECIAL CYCLIC SYSTEM*

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1. Introduction. This note is concerned with a special cyclic system.† Let $S$ be a surface referred to any orthogonal system, and $T$ the trihedral whose $x$-axis is tangent to the curve $v = \text{const}$. The equations

$$x = R(1 + \cos \theta), \quad y = 0, \quad z = R \sin \theta,$$

define a two-parameter family of circles $C$ normal to $S$; and the necessary and sufficient conditions that $C$ shall constitute a cyclic system are

$$\frac{\partial R}{\partial v} + R \eta_1 r = 0, \quad R(p r_1 - p_1 r) - q_1 \left( \xi + \frac{\partial R}{\partial u} \right) + q \frac{\partial R}{\partial v} = 0.$$

It is readily seen that the first of equations (2) may be written‡

$$\xi \frac{\partial R}{\partial v} = R = U \xi,$$

hence

$$R = U \xi,$$

where $U$ is a function of $u$ alone. Using (3) we may write the second equation of (2) in the form

$$U \xi (p r_1 - p_1 r) - q_1 \left( \xi + \xi U' + U \frac{\partial \xi}{\partial u} \right) - q \eta_1 r = 0.$$

We shall therefore replace equations (2) by (3) and (4).

2. The Inversion of $C$. If we invert the circles $C$ relative to the circles $x^2 + z^2 = K^2$, $y = 0$, where $K$ is any constant, we get the following system of lines $L$,

$$x = \frac{K^2}{2R}, \quad y = 0,$$

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† See Eisenhart, Differential Geometry of Curves and Surfaces, Ex. 11, p. 444.
‡ Eisenhart, p. 170.
which will constitute a rectilinear congruence. Let us determine the condition that this congruence be normal.

Any point on the lines (5) will have the coordinates \((K^2/(2R), 0, z)\). The necessary and sufficient condition that (5) shall define a normal congruence is that \(\delta z = 0\) for all values of \(dv/du\). Hence

\[
\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{K^2}{2R} (qdu + q_1dv) = 0,
\]

or

\[
\frac{\partial z}{\partial u} = \frac{K^2q}{2R}, \quad \frac{\partial z}{\partial v} = \frac{K^2q_1}{2R}.
\]

The condition of integrability is that \(\partial^2 z/\partial u \partial v = \partial^2 z/\partial v \partial u\); hence

\[
\frac{\partial q}{\partial v} - \frac{\partial R}{\partial v} = R \frac{\partial q_1}{\partial u} - q_1 \frac{\partial R}{\partial u},
\]

which by means of (3) becomes

\[
U\xi (rp_1 - r_1p) + qU \eta r + q_1 \left( \xi U' + U \frac{\partial \xi}{\partial u} \right) = 0.
\]

On making use of (4), this further reduces to \(q_1 = 0\); hence we have the following theorem.

**Theorem 1.** A necessary and sufficient condition that the lines \(L\), obtained by the above inversion of the cyclic system \(C\), shall form a normal congruence, is that \(S\) be referred to its lines of curvature.

We readily find that for \(q_1 = p = 0\), (4) becomes \(r (\xi p_1 + \eta q) = 0\). Two cases are to be considered.

(a) If \(r = 0\), the curves \(v = \text{const.}\) are geodesics; and since they are lines of curvature, they are plane curves. Consequently the planes of \(C\) constitute but a one-parameter family, and their envelope is the developable surface which is readily seen to be one of the focal sheets of \(S\). We note also from (2) that for \(r = 0\), \(R\) is a function of \(u\) alone.

(b) If \(\xi p_1 + \eta q = 0\), it is readily seen\(^\dagger\) that \(D: E = D'' : G\). Hence \(S\) is in this case either a sphere or a plane. If \(S\) is a sphere the planes of \(C\) pass through the center.\(^\ddagger\)

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* Eisenhart, p. 168.
† Eisenhart, p. 174.
‡ Eisenhart, p. 441.
3. *The Focal Points of C.* The direction cosines of an arbitrary tangent of \( C \) are \((-\sin \theta, 0, \cos \theta)\). Since the displacements \( \delta x, \delta y, \delta z \) of the focal points of \( C \) must be in the direction of the tangents we have \( \cos \theta \delta x + \sin \theta \delta z = 0, \delta y = 0 \). On using (1), these equations may be written

\[
\frac{x}{R} \frac{\partial R}{\partial u} \text{du} + \frac{x}{R} \frac{\partial R}{\partial v} \text{dv} + \frac{\xi(x - R)}{R} \text{du} - z(qdu + q_1dv) = 0,
\]

\[
\eta_1dv + x(rdu + r_1dv) = 0.
\]

Obviously, we must have

\[
\left| \begin{array}{cc}
\frac{x}{R} \frac{\partial R}{\partial u} + \frac{\xi(x - R)}{R} & x \frac{\partial R}{\partial v} - q_1z \\
r_1x - p_1z & \eta_1 + r_1x - p_1z
\end{array} \right| = 0.
\]

Hence the focal points of \( C \), which are at most four in number, are given by (1) and (7).

Let us now consider the case when the inversion of \( C \) leads to a normal rectilinear congruence. We have seen that we must have \( p = q_1 = 0, \) and \( r(\xi p_1 + \eta_1q) = 0 \). The first case, (a), is of some interest; for \( p = q_1 = r = 0, \) equation (7) becomes*

\[
\frac{\partial R}{\partial u} + \xi \left( x - Rqz - R\xi = 0, \quad r_1x - p_1z + \eta_1 = 0. \right.
\]

We note that the two centers of principal curvature, \((0, 0, -\xi/q)\) and \((0, 0, \eta_1/p_1)\), lie in the planes (8), while the second plane of (8) also contains the center of geodesic curvature, \((-\eta_1/r_1, 0, 0)\), of the curve \( u = \text{const} \). Hence we have the following theorem.

**Theorem 2.** *If the above inversion of \( C \) leads to a normal rectilinear congruence, and the curves \( v = \text{const} \) are geodesics, \((r = 0)\), then two of the focal points of \( C \) are collinear with the center of principal curvature of the curve \( v = \text{const} \), and the other two are collinear with the centers of principal and geodesic curvature of the curve \( u = \text{const} \).*

From (6) it is evident that for \( p = q_1 = r = 0, \) and \( R = f(u) \), \( du \) and \( dv \) are factors of these equations. Hence we have the following theorem.

* \( R \) is a function for \( u \) alone for (a).
Theorem 3. If the above inversion of $C$ leads to a normal rectilinear congruence, and the curves $v = \text{const.}$ are geodesics, then the circles of $C$ which have an envelope are those which correspond to the lines of curvature on $S$.

4. The Focal Points of $L$. The Developables of $L$. As the vertex of $T$ is displaced along the curves of $S$ which define the developables of $L$, the displacements $\delta x$ and $\delta y$ of the focal points will be zero. Hence from (5), when $L$ is a normal congruence, we have

$$
\frac{K^2}{2R^2} \left( \frac{\partial R}{\partial u} du + \frac{\partial R}{\partial v} dv \right) - \xi du - zqdu = 0,
$$

(9)

$$
\eta_1 dv + \frac{K^2}{2R} (r du + r_1 dv) - zp_1 dv = 0.
$$

The elimination of $dv/du$ between equations (9) gives us a quadratic in $z$ whose roots determine the focal points of $L$; the elimination of $z$ gives the equation of the curves on $S$ defining the developables of $L$.

For (a), that is for $r = 0, R = f(u)$, the focal points of $L$ as determined by (9) are

$$
z_1 = \frac{1}{q} \left( \frac{K^2}{2R^2} \frac{\partial R}{\partial u} - \xi \right), \quad z_2 = \frac{1}{\rho_1} \left( \frac{K^2 r_1}{2R} + \eta_1 \right).
$$

(10)

When we put (5) in the second of equations (8) and solve for $z$ we get the value of $z_2$ in (10). Hence one of the focal points of $L$ is collinear with the two focal points of $C$ determined by the second member of (8). It is readily shown from (5), (8), and (10) that the other focal point of $L$ is not collinear with the other two focal points of $C$.

From (9) it is readily seen that for (a), $du$ and $dv$ are factors of these equations. Hence we have the following theorem.

Theorem 4. If the above inversion of $C$ leads to a normal rectilinear congruence $L$, and the curves $v = \text{const.}$ are geodesics, the lines of curvature on $S$ define both the developables of $L$, and those families of $C$ which have an envelope.