ON THE NUMBER OF \((q+1)\)-SECANT \(S_{q-1}\)'S OF A CERTAIN \(V_{k}^{n}\) IN AN \(S_{qk+q+k-1}\)

BY B. C. WONG

In this note we are concerned only with those \(k\)-dimensional non-developable varieties which are rational loci each of \(\infty^{1}(k-1)\)-spaces. By a rational locus of \(\infty^{1}(k-1)\)-spaces we mean one whose \((k-1)\)-spaces can be put in a one-to-one correspondence with the points of a straight line. Let such a locus or variety, \(V_{k}^{n}\), of order \(n\) be given in an \(S_{r}\). Now in \(S_{r}\) there are \(\infty^{1}(r-q+1)(q-1)\)-spaces. For a \((q-1)\)-space to meet \(V_{k}^{n}\) \(q+1\) times is equivalent to \((q+1)(r-q-k+1)\) simple conditions. In order that the number, \(N\), of \((q-1)\)-spaces \((q+1)\)-secant to \(V_{k}^{n}\), that is, having \(q+1\) points of simple incidence with \(V_{k}^{n}\), be finite, we must have \((q+1)(r-q-k+1) = q(r+q+1)\) or \(r=qk+q+k-1\). It is our purpose to determine this number \(N\) of \((q+1)\)-secant \(S_{q-1}\)'s of \(V_{k}^{n}\) in \(S_{qk+q+k-1}\).

For this purpose we find it convenient to consider the \(V_{k}^{n}\) in question as the projection of a \(V_{k}^{n}\) in a higher space \(S_{r'}\). This \(V_{k}^{n}\) may always be regarded as the locus of \(\infty^{1}(k-1)\)-spaces joining corresponding points of \(k\) rational, projectively related curves \(C_{n1}, C_{n2}, \ldots, C_{nk}\) of respective orders \(n_{1}, n_{2}, \ldots, n_{k}\), where \(n_{1}+n_{2}+\cdots+n_{k}=n\). The \(S_{r'}\) containing \(V_{k}^{n}\) must be such that \(r' \leq n+k-1\). If \(r'=n+k-1\), \(V_{k}^{n}\) is said to be normal in \(S_{n+k-1}\). It is only necessary to consider this normal \(V_{k}^{n}\).

Let the \(k\) curves be given parametrically by

\[
\begin{align*}
C_{n1} & \quad x_{0}:x_{1}: \cdots :x_{n_{1}} = t^{n_{1}}:t^{n_{1}-1}: \cdots :1, \\
& \quad x_{n_{1}+1} = x_{n_{1}+2} = \cdots = x_{n_{1}+k-1} = 0; \\
& \quad x_{n_{1}+1}:x_{n_{1}+2}: \cdots :x_{n_{1}+n_{2}+1} = t^{n_{2}}:t^{n_{2}-1}: \cdots :1, \\
& \quad x_{n_{1}+n_{2}+2} = x_{n_{1}+n_{2}+3} = \cdots = x_{n_{1}+k-1} = 0; \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
C_{nk} & \quad x_{0} = x_{1} = \cdots = x_{n_{k}+k-2} = 0, \\
& \quad x_{n_{k}+k-1}:x_{n_{k}+k} : \cdots :x_{n_{k}+k-1} = t^{n_{k}}:t^{n_{k}-1}: \cdots :1.
\end{align*}
\]
Then a general point of $V_k^n$ has the coordinates

$$(\lambda_1 t_0^{x_0}; \lambda_1 t_0^{x_1}; \cdots; \lambda_1; \lambda_2 t_0^{x_0}; \lambda_2 t_0^{x_1}; \cdots; \lambda_2; \cdots; \lambda_h t_0^{x_0}; \lambda_h t_0^{x_1}; \cdots; \lambda_h).$$

Now let $t$ take on $q+1$ values, say $t_0, t_1, \cdots, t_q$, and we have $q+1$ points on $V_k^n$ determining an $S_q$. The parametric equations of this $S_q$ are, the parameters being the $l'$s,

$$x_{n-k+1} = \lambda_l \sum_{i=0}^{q} (l_i t_0^{x_i} - j_i),$$

$$(k = 1, 2, \cdots, k; j_i = 1, 2, \cdots, n).$$

If we now eliminate the $t$'s, $l$'s, and $\lambda$'s from the above equations of $S_q$, we obtain

$$x_{n-k+1} + \cdots + x_{n-k+q} = \lambda_i \sum_{i=0}^{q} (l_i t_0^{x_i} - j_i) = 0.$$
$M$ points. Now let both $V_k'^n$ and $V_q^n$ be projected from $S_{n-q+k-1}$ upon an $S_{q+k+q+k}$. The projection of the former is a $V_k'^n$ and that of the latter is a system of $\infty^{(q+1)}$ $q$-spaces. Each of these $q$-spaces is $(q+1)$-secant to $V_k'^n$ and $M$ of them pass through a given point $P$. If we now project $V_k''^n$ from $P$ upon an $S_{q+k+q+k-1}$ of $S_{q+k+q+k}$, we obtain for projection the $V_k^n$ the number $N$ of whose $(q+1)$-secant $(q-1)$-spaces we wish to find. The $(q+1)$-secant $S_{q-1}$'s of $V_k^n$ are the $(q-1)$-spaces in which $S_{q+k+q+k-1}$ intersects the $(q+1)$-secant $S_q^n$'s of $V_k''^n$ passing through $P$. Hence the number $N$ we are seeking is equal to $M$, that is,

$$N = \binom{n-qk}{q+1}.$$  

Thus, for $k=1$, we have a rational curve $C^n$ in $S_{2q}$ having $\binom{n-1}{q+1}$ $(q+1)$-secant $S_{q-1}$'s. If $q=1$, we have the familiar case of a rational plane curve of order $n$ with $(n-1)(n-2)/2$ double points. If $q=2$, we have the case which is also familiar of a rational 4-space curve having $(n-2)(n-3)(n-4)/6$ trisecant lines.

Let $k=2$ and we have a rational ruled surface $F^n$ of order $n$ in $S_{3q+1}$ with $\binom{n-2q}{q+1}$ $(q+1)$-secant $S_{q-1}$'s. Thus, a rational $F^n$ in $S_4$ has $(n-2)(n-3)/2$ improper double points; an $F^n$ in $S_7$ has $(n-4)(n-5)(n-6)/6$ trisecant lines.

If we put $k=3$ and then $q=1, 2, 3, \cdots$, successively, we find, by what precedes, that a rational planed variety $V_3^n$ of order $n$ in $S_6$, $S_{10}$, $S_{14}, \cdots$, has, respectively, $(n-3)$ $(n-4)/2$ improper double points, $(n-6)(n-7)(n-8)/6$ trisecant lines, $(n-9)(n-10)(n-11)(n-12)/24$ quadrisecant planes, $\cdots$.

The University of California