A MODULAR MANIFOLD ASSOCIATED WITH THE GENERALIZED KUMMER MANIFOLD \((p = 3)\)

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1. Introduction. The generalized Kummer \(p\)-way manifold, \(K_p\), is determined by equating the homogeneous coordinates of a point \(P\) in \(S_{2p-1}\) to linearly independent theta functions of the second order and zero characteristic.* As the variables \(u\) in these functions change, the point \(P(u)\) runs over the manifold \(K_p\) in \(S_{2p-1}\). If the variables \(u\) be increased by a half-period \(\pi\), the point \(P(u)\) is transformed into the point \(P(u') = P(u + \pi)\). Thus the half-periods determine a group \(G_{2p}\) of birational transformations of \(K_p\) into itself. Klein and Wirtinger have shown that these birational transformations are effected on \(K_p\) by the operations of a collineation group \(G_{2p}\) in \(S_{2p-1}\) under which \(K_p\) is invariant. The functions which define the position of \(P\) may be so chosen that the coefficients of the collineations of \(G_{2p}\) are numerical. A convenient choice† is that of functions \(Z_{\eta_1} \ldots Z_{\eta_p}\), \((\eta_i = 0, 1)\), for which the addition of a particular half-period \(\pi_{\eta_i}\)

* The reader is referred to the following sources.
† Coble, Colloquium, loc. cit., p. 94.
changes $Z_{\eta_1} \ldots Z_{\eta_p}$ into $(-1)^{Z_p} Z_{\eta_1} \ldots Z_{\eta_p}$; and for which the addition of another particular half-period $\pi_{\eta_i}$ interchanges the values 0, 1 of the index $\eta_i$ of $Z_{\eta_1} \ldots Z_{\eta_p}$ ($i = 1, \ldots, p$). The $2^{2p}$ points $P(u)$ for which $u = \pi$ are singular points of $K_p$. A particular $K_p$ is determined when the group $G_{2p}$ of $K_p$, and a singular point of $K_p$, are given.

When $G_{2p}$ is fixed, say in the simple form just indicated, there is a family, $F_p$, of $K_p$'s each of which admits this group. This family is obtained by variation of the moduli $a_{ij}$ of the theta functions. As $K_p$ runs through this family $F_p$, the $2^{2p}$ singular points of $K_p$ describe a locus in $S_{2p-1}$, the modular manifold $M$, with which we shall be concerned for the case $p = 3$. In the case $p = 1$, $K_p$ is a doubly covered $S_1$ with four branch points as singular points. As the modulus $a$ of the elliptic thetas changes, these branch points run over the entire $S_1$. In the case $p = 2$, $K_p$ is the Kummer surface in $S_3$ with 16 nodal singular points. As the three moduli $a_{ij}$ change, these nodes run over the entire $S_3$. In the case $p = 3$, $K_p$ is the Kummer 3-way in $S_7$ of order 24 with 64 four-fold singular points. In this case, however, as the six moduli $a_{ij}$ change, the singular points in $S_7$ run over a manifold of dimension six and order 16, $M_{6}^{16}$. The purpose of this article is to discuss this manifold $M_{6}^{16}$ which appears from the transcendental viewpoint, with respect to its projective and group-theoretic properties. We use without further explanation the standard notations of the theta function theory. The equation of $M_{6}^{16}$ is obtained in §2. In §§3–4, the multiplicities of certain loci on $M_{6}^{16}$ and on its sections are determined. In §5 a projective determination of $M_{6}^{16}$ by means of these multiplicities is indicated.

2. Determination of the Equation of $M_{6}^{16}$. The equation of $M_{6}^{16}$ may be obtained from one of the relations existing between products of the zero values of the even thetas. One such is

$$\theta_{1mn7} \theta_{1mn8} \theta_{i7n8} \theta_{i7n8} + \theta_{1mn7} \theta_{1mn8} \theta_{i7n8} \theta_{i7n8} \pm \theta_{kmn7} \theta_{kmn8} \theta_{ikn7} \theta_{ikn8} = 0.$$  

This may be written, in a particular case, as

$$\left( \sum_{\alpha=1}^{3} \pm (c_{\alpha}^{457} c_{\alpha}^{458} c_{\alpha}^{457} c_{\alpha}^{468})^{1/2} = 0, \right.$$

(1)
where \( \theta_{ijk}(0) = c_{ijk} \). To obtain the expression for this in terms of the modular functions \( z_{ijk} = z_{ijk}(0) \), it is convenient to write the relation as \( (a)^{1/3} + (b)^{1/3} + (c)^{1/3} = 0 \). The rational form is \( c^2 - 2c(a + b) + (a - b)^2 = 0 \). In expressing the \( c_{ijk} \) in terms of the \( z_{ijk} \)'s it is convenient to set \( z_{000} = z, \) and the remaining \( z_{ijk} = z_1, z_2, \cdots, z_7 \), where the \( z_1, z_2, \cdots, z_7 \) are thought of as attached to the seven points of a finite planar geometry, mod 2, say \( PG(2, 2) \), in particular \( z_{ijk} \) attaching to the point with co-ordinates \( i, j, k = 0, 1 \). Then the \( a, b, c, \) above, become

\[
a = [(z^2 - z_0^2) + (z_1^2 - z_2^2) + (z_3^2 - z_4^2) + (z_5^2 - z_6^2)]
\cdot [(z^2 - z_0^2) + (z_1^2 - z_2^2) - (z_3^2 - z_4^2) + (z_5^2 - z_6^2)]
\cdot [(z^2 - z_0^2) + (z_1^2 - z_2^2) - (z_3^2 - z_4^2) - (z_5^2 - z_6^2)]
\cdot [(z^2 - z_0^2) - (z_1^2 - z_2^2) + (z_3^2 - z_4^2) - (z_5^2 - z_6^2)]
\cdot [(z^2 - z_0^2) - (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) - (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) - (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) - (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) - (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)],
\]

\[
b = [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) - (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) - (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) - (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) - (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)]
\cdot [(z^2 + z_0^2) + (z_1^2 + z_2^2) + (z_3^2 + z_4^2) + (z_5^2 + z_6^2)],
\]

\[
c = [2z_5z_6 + 2z_1z_2 + 2z_3z_7 + 2z_4z_8] \cdot [2z_5z_6 + 2z_1z_2 - 2z_3z_7 - 2z_4z_8]
\cdot [2z_5z_6 - 2z_1z_2 - 2z_3z_7 - 2z_4z_8] \cdot [2z_5z_6 - 2z_1z_2 + 2z_3z_7 + 2z_4z_8].
\]

The rationalized form then yields the following equation for \( M_{616} \):

\[
M_{616} = 2z_1z_2z_3z_4z_5z_6z_7 \left\{ -2z^8 + 4z^4 \sum_{7} z_1^4 - 16z^2 \sum_{7} z_1^2 z_2^2 z_3^2 \right\}
- 2 \sum_{7} z_1^8 - 16 \sum_{7} z_1^4 z_2^2 z_3^2 z_4^2 + 4 \sum_{21} z_1^4 z_2^4 \right\}
+ z^8 \left( \sum_{7} z_1^2 z_2^2 z_3^2 z_4^2 z_5^2 \right)
+ z^4 \left( - \sum_{7} z_1^4 z_2^2 z_3^2 z_4^2 z_5^2 \right)
+ z^4 \left( - \sum_{7} z_1^4 z_2^2 z_3^2 z_4^2 z_5^2 + 2 \sum_{7} z_1^4 z_2^2 z_3^2 z_4^2 z_5^2 + \sum_{28} z_1^4 z_2^4 z_3^4 \right)
+ z^2 \left( \sum_{7} z_1^2 z_2^2 z_3^2 z_4^2 z_5^2 \right) - \sum_{7} z_1^4 z_2^2 z_3^2 z_4^2 z_5^2 + 2 \sum_{7} z_1^4 z_2^4 z_3^2 z_4^2 z_5^2
\]
This final form of the equation of $M_{6}^{16}$ contains first the odd and then the even powers of $z$ in descending order. The symbol $\sum_{7} z^{3} z^{2} z^{2} z^{2} z^{2} z^{2} z^{2}$ represents the sum of seven terms whose three subscripts lie on a line in the finite geometry, $PG(2, 2)$. Other symbols in (2) refer to the seven quadrilaterals, seven points with each of four outside lines, seven points with each of three outside quadrangles, twenty-eight triangles, and twenty-one pairs of intersecting lines. Thus the modular locus is of order 16, with a 7-fold point at each reference point with a tangent septimic cone made up of the seven reference $S_{0}$'s on the 7-fold point, and with similar behavior at the conjugates of these under the group of $M_{6}^{16}$.

This modular manifold, $M_{6}^{16}$, is invariant† under a group $G$ of order $8! \cdot 36 \cdot 64$ generated by elements $J_{abc}, l_{mn}$. The situation is given by the following statement.

**Theorem 1.** $M_{6}^{16}$ is invariant under a group $G$ of order $8! \cdot 36 \cdot 64$. This group $G$ is the largest collineation group which contains the collineation group $G_{64}$ of $K_{3}^{24}$ as an invariant subgroup.

Corresponding to the fact that the seven points of a finite plane (mod 2) may be permuted in 168 ways without destroying the linearity of triads, the equation of $M_{6}^{16}$ is invariant under a permutation group of $z_{1}, \cdots, z_{7}$ of order 168, a subgroup of $G$. There exists a larger permutation subgroup $G_{8,168}$ on $z, z_{1}, \cdots, z_{7}$. Under the symmetry imposed by this larger subgroup in the equation (2) of $M_{6}^{16}$ is reduced to the form

$$M_{6}^{16} = \left\{ \prod_{8} (z) \right\}^{2} + 2 \sum_{8} z^{8} + 4 \sum z^{4} z_{1}^{4} - 16 \sum_{14} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2}$$

$$+ \sum_{7} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2}$$

$$+ \sum_{12} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2} z_{8}^{2}$$

$$+ 72 \left\{ \prod_{8} (z) \right\}^{2} = 0.$$
3. **Multiple Loci of $M_6^{16}$.** A. 7-fold points. The reference octahedron is one of a set of 135 octahedra all of which are conjugate under $G$. A particular octahedron is associated with a Göpel system of 7 mutually syzygetic half periods. No two of these octahedra have a vertex in common, whence the number of 7-fold points of $M_6^{16}$ is $8 \cdot 135 = 1080$.

B. 4-fold $S_3$'s. To each half-period of the theta functions there is associated an element $J$ of period four whose square $I$ is in $G_{64}$. As an involutorial element in $G_{64}$, $I$ has a locus of fixed points made up of the two skew $S_3$'s. Thus there is a correspondence between the 63 half-periods and 63 pairs of skew $S_3$'s. A typical pair of $S_3$'s is $z = s_1 = s_2 = s_6 = 0$, $z_3 = z_4 = z_5 = z_7 = 0$. From the form of (2), we have the following theorem.

**Theorem 2.** Each of the 126 $S_3$'s determined by the 63 half-periods lies on $M_6^{16}$ and is a 4-fold locus of it.

The sum of two half-periods is a third. Such a linearly related triad of half-periods will contain pairs which are either (a) syzygetic or (b) azygetic. We examine the intersections of the three pairs of $S_3$'s determined by such syzygetic or azygetic triads. The number of such triads is 315 or 336 as the case may be.*

In case (a) as it appears for $p = 2$, the syzygetic triad of half-periods determines three pairs of lines which are the three pairs of opposite edges of a 4-point in $S_3$. For $p = 3$, however, they determine three pairs of $S_3$'s which are the three pairs of opposite $S_3$'s on a 4-line $\lambda$ in $S_7$. Thus each line $\lambda$ is on three $S_3$'s. It is a line joining two vertices of one of the 135 reference octahedra and each line $\lambda$ is on three pairs of such vertices. These 1260 lines $\lambda$ are an extension of the 60 points of Klein's 60$_{15}$ configuration ($p = 2$). We may state the following result.

**Theorem 3.** The 126 $S_3$'s determined by the 63 half-periods intersect by threes in 1260 lines $\lambda$ which are 6-fold lines of $M_6^{18}$. These lines constitute one extension of Klein's 60$_{15}$ configuration ($p = 2$), the other being the 135 reference octahedra. The 30 lines $\lambda$ in each $S_3$ form the edges of a 60$_{18}$ configuration ($p = 2$).

* Coble, Colloquium, loc. cit., §33.
Thus a plane through three vertices of a reference octahedron in $S_7$ cuts the manifold in an 18-ic, and therefore lies on the manifold. Any one of these planes is a 4-fold locus on $M_6^{16}$.

In case (b), and for $p = 2$, the azygetic triad of linearly related half-periods determines three pairs of lines which are generators of one ruling of the quadric in $S_3$ which is determined by that even theta function which has the three half-periods as zeros. For $p = 3$, they determine three pairs of $S_3$'s any two of which are skew to each other. These six $S_3$'s are $S_3$-generators of one ruling* for each of the quadrics in $S_7$ determined by each of the six even theta functions which has the three half-periods as zeros.

4. Linear Sections of $M_6^{16}$. We examine only those linear sections of $M_6^{16}$ which are most effective for its projective determination. We have already noted that the $S_3$ of the type $z = z_1 = z_2 = z_3 = 0$ lies on $M_6^{16}$. On the other hand, from the equation of $M_6^{16}$, we have the following theorem.

**Theorem 4.** An $S_3$ of the type $z = z_1 = z_2 = z_3 = 0$ cuts $M_6^{16}$ in the four faces of a tetrahedron each repeated four times.

We shall also need the sections $M_4^{16}$ of $M_6^{16}$ by $S_5$'s of the type $z = z_1 = 0$. Each of the 135 reference octahedra determines 28 of these $S_5$'s but each $S_5$ occurs in three octahedra, whence there are 45 · 28 of these $S_5$'s and each is invariant under a subgroup $G_{3,48,2^2}$ of $G$. The factor 3 of this order is due to the three octahedra which contain the $S_5$; the factor 48 is the order of the subgroup of permutations of $z, z_1, z_2, z_3$ which leaves the pair $z, z_1$ unaltered; and the factor 2² represents the multiplicative subgroup of $z, z_1, z_2, z_3$ which appears in $G$.

Such an $S_5$ is cut by the 63 pairs of $S_5$'s of Theorem 2 in pairs of linear spaces of the following character: $(\alpha)$ 3 consisting of an $S_3$ and an $S_1$; $(\beta)$ 12 consisting of pairs of $S_5$'s; $(\gamma)$ 48 consisting of pairs of $S_1$'s. According to Theorem 2 these are four-fold loci on $M_4^{16}$.

**Theorem 5.** An $M_4^{16}$ cut out on $M_6^{16}$ by an $S_5$ of type $z = z_1 = 0$ has the loci $(\alpha), (\beta), (\gamma)$ and the faces described in Theorem 4 as four-fold loci.

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* Bertini, loc. cit., p. 143.
The $S_i$'s of $(\alpha)$ are actually 8-fold loci on $M_4^{16}$. It is now relatively easy to prove the following theorem.

**Theorem 6.** $M_4^{16}$ is the manifold of lowest order in $S_5$ invariant under $G_{48,3,2^9}$ with the multiplicities described in Theorem 5.

For its invariance under the multiplicative group of order $2^9$ rules out all terms whose exponents do not satisfy certain congruences. Its invariance under the permutation group of order 48 mentioned above necessitates the equality of various sets of coefficients. The remaining indeterminations are then easily removed by applying the multiplicity conditions of Theorems 4, 5.

5. Determination of a Unique Manifold of Lowest Order with Multiplicities of §3, Invariant under the Group $G$ of Order $8! \cdot 36 \cdot 64$. The original form of the equation of $M_6^{16}$ was obtained from transcendental considerations. But the invariance of $M_6^{16}$ under the multiplicative group of order $2^9$, $(J_{abc,1mn})$, is sufficient to determine the type of terms which may appear in its equation. This is accomplished by the solution of congruences. Its invariance under the permutation group $G_{8,168}$ of $z, \cdots, z_7$ reduces the number of unknown coefficients to 8. These coefficients may now be determined by applying Theorem 6 to the $S_6$'s defined by the reference octahedron. Thus a characteristic projective property of the manifold, $M_6^{16}$, originally defined by considerations in which function theory is essential, has been obtained. We may then make the following statement.

**Theorem 7.** $M_6^{16}$ is the manifold of lowest order in $S_7$ invariant under the group $G$ of order $8! \cdot 36 \cdot 64$ with the multiplicities of §3.