INVOLUTORIAL LINE TRANSFORMATIONS
DETERMINED BY CREMONA PLANE INVOLUTIONS*

BY J. M. CLARKSON

1. Introduction. The author has discussed† an involutorial line transformation effected by considering a harmonic homology in each of two planes. If \( A, B \) be the points in which an arbitrary line \( (y) \) meets the planes \( \alpha, \beta \), and if \( A', B' \) be their images by the homologies \( I_\alpha, I_\beta \), respectively, then \( (x) \equiv A'B' \) is the transform of \( (y) \). It is the purpose of the present paper to consider the line transformations similarly determined by Cremona involutorial transformations in each of two planes. All combinations of the four fundamental types: Homology; Jonquières; Geiser; and Bertini will be considered. The orders of the transformations, the invariant loci, the singular elements and the transforms of certain elementary forms are discussed.

2. Homology-Jonquières. In the plane \( \alpha \) consider a harmonic homology \( I_\alpha \), center at \( O_1 \) and axis \( \Delta_\alpha \). In the plane \( \beta \) consider the perspective Jonquières involution \( I_\beta \), of order \( n \), with basis point \( P_1 \) of multiplicity \( (n-1) \) and basis points \( P_2, \ldots, P_{2n-1} \) each simple, and with invariant curve \( \Delta_\beta: P_1^{n-2} P_2^1 \ldots P_{2n-1}^1 \) of order \( n \) and genus \( (n-2) \).

An arbitrary line \( (y) \) meets \( \alpha \) in a point \( A \) whose coordinates are linear in the Plücker coordinates \( y_i \) of \( (y) \) and meets \( \beta \) in a point \( B \) whose coordinates are also linear in \( y_i \). The image \( A' \) of \( A \) by \( I_\alpha \) has coordinates also linear in \( y_i \) but the image \( B' \) of \( B \) by \( I_\beta \) has coordinates which are functions of degree \( n \) in \( y_i \). Hence \( (x) \equiv A'B' \) has Plücker coordinates of degree \( (n+1) \) in \( y_i \). Thus, the transformation

\[
(1) \quad x_i = \phi_i(y)
\]

is of order \( (n+1) \). The invariant lines of (1) form a congruence \((n, n)\) composed of the lines meeting \( \Delta_\alpha, \Delta_\beta \); and in addition there is a cone of order \( n \), vertex \( O_1 \), base curve \( \Delta_\beta \).

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If \((y)\) meets \(\beta\) in \(P_1\) and \(\alpha\) in \(A\), then \((y) \sim\) a cone of order \((n-1)\), base curve the curve of order \((n-1)\) into which \(P_1\) is transformed by \(I_\beta\). If \((y)\) meets \(\beta\) in \(P_i(i \neq 1)\) and \(\alpha\) in \(A\), then \((y) \sim\) the pencil whose vertex is \(A'\) and in the plane \(A'P_iP_i\).

The image by \(I_\alpha\) of the line \(c \equiv \alpha \beta\) is a line \(c_\alpha\) and by \(I_\beta\) is a curve \(c_\beta\) of order \(n\). The points \(C_\alpha\) of \(c_\alpha\) and \(C_\beta\) of \(c_\beta\) are projective with the points \(C\) of \(c\). Hence the line joining any two corresponding points \(C_\alpha\), \(C_\beta\) is transformed by (1) into the bundle whose vertex is \(C\). These lines \(C_\alpha C_\beta\) form a regulus \(\{c\}_{n+1}\) of order \((n+1)\), every generator of which is singular.

A line \(t_\alpha\) in \(\alpha\) meets \(\beta\) in a point \(C\) whose image by \(I_\beta\) is \(C_\beta\). The image of \(t_\alpha\) by \(I_\alpha\) is a line \(t'_\alpha\). Hence \((y) = (t_\alpha) \sim\) a pencil, vertex \(C_\beta\), plane \(C_\beta t'_\alpha\). If \(t_\alpha\) pass through \(O_1\), then \(t'_\alpha \equiv t_\alpha\) and the plane of the pencil is \(C_\beta t_\alpha\). If \(t_\alpha\) be the axis \(\Delta_\alpha\), then \(t'_\alpha \equiv \Delta_\alpha\) and the plane of the pencil is \(C_\beta \Delta_\alpha\).

A line \(t_\beta\) in \(\beta\) meets \(\alpha\) in a point \(C\) whose image by \(I_\alpha\) is \(C_\alpha\). The image of \(t_\beta\) by \(I_\beta\) is a curve \(\rho_\beta\) of order \(n\). Hence \((y) = (t_\beta) \sim\) a cone of order \(n\), vertex \(C_\alpha\), base curve \(\rho_\beta\). If \(t_\beta\) pass through \(P_i\), then \(\rho_\beta\) is a line, and indeed the line \(t_\beta\). Hence the conjugate is no longer a cone but a pencil, vertex \(C_\alpha\), plane \(C_\alpha t_\beta\). If \(t_\beta\) pass through \(P_i(i \neq 1)\), then \(\rho_\beta\) is of order \((n-1)\) and the conjugate cone is of order \((n-1)\). If \(t_\beta\) be the line \(P_1P_i\), then \((y) = (t_\beta) \sim C_\alpha P_i\).

As \(t_\alpha\) describes the pencil \((C, \alpha)\) each conjugate pencil has its vertex at \(C_\beta\) and its plane passes through \(C_\alpha\). As \(C\) describes \(c\), \(C_\beta\) describes \(c_\beta\), and hence the plane field \((\alpha) \sim\) the special complex of order \(n\) with \(c_\beta\) as directrix curve.

As \(t_\beta\) describes the pencil \((C, \beta)\), then each conjugate cone has its vertex at \(C_\alpha\) and has \(C_\alpha C_\beta\) as a generator. As \(C\) describes \(c\), \(C_\alpha\) describes \(c_\alpha\), and hence the plane field \((\beta) \sim\) the special linear complex \(|c_\alpha|\).

Each line \(t_\alpha\) of a pencil \((T, \alpha)\) is transformed by (1) into a pencil whose vertex is on \(c_\beta\) and whose plane passes through \(t'_\alpha\) which belongs to the pencil \((T', \alpha)\), \(T'\) being the image by \(I_\alpha\) of \(T\). These \(\alpha \sim\) pencils form a congruence \((n+1, n)\). If a line \((x)\) meet \(c_\beta\) in a point \(C_\beta\) and also meet the \(t'_\alpha\) which corresponds to \(C_\beta\), then \((x)\) belongs to the conjugate congruence. Through an arbitrary point of space a line \((x)\) belonging to the conjugate congruence has coordinates which are of degree \((n+1)\) in the parameter \(\lambda\) of a line of \((T, \alpha)\). Hence the order of the con-
An arbitrary plane meets \( c_2 \) in \( n \) points and meets each corresponding \( t_d' \) in one point. Hence the class is \( n \).

Likewise a pencil \( (T, \beta) \sim \) a congruence \( (n+1, n) \). However, if \( T \) lie at \( P_i \), the order and class are both reduced so that the congruence is \( (2, 1) \). If \( T \) lie at \( P_i(i \neq 1) \), then the congruence is \( (n, n-1) \).

An arbitrary pencil \( (T, \tau) \sim \) a regulus \( R \) of order \( (n+1) \), the generators of which are the joins of corresponding points on the straight line image by \( I_\alpha \) of \( \tau \alpha \) and the curve of order \( n \), image by \( I_\beta \) of \( \tau \beta \). If \( \tau \) pass through \( P_i \), then \( R \) is of order 2; through \( P_i(i \neq 1) \), of order \( n \); through \( P_iP_j(i, j \neq 1) \), of order \( (n-1) \); through \( P_iP_i \), the conjugate is a pencil with vertex \( P_i \).

An arbitrary plane field of lines \( (\tau) \sim \) a congruence \( (n, n) \) composed of lines meeting a line and a plane curve of order \( n \) not meeting the line. If \( \tau \) pass through \( P_i \), the congruence is \( (1, 1) \); through \( P_i(i \neq 1) \), \( (n-1, n-1) \); through \( P_iP_j(i, j \neq 1) \), \( (n-2, n-2) \); through \( P_iP_i \), the conjugate is no longer a congruence but a pencil, vertex \( P_i \).

An arbitrary bundle \( (T) \sim \) a congruence \( (3n, n) \). From an arbitrary point of space, the points of the planes \( \alpha, \beta \) form two projective fields. There are three coincidences in a section of such a projection. Hence the parameters \( \lambda, \mu \) of a line of \( (T) \) appear to degree \( 3n \) in defining a line \( (x) \) of the conjugate congruence through an arbitrary point of space. In an arbitrary plane of space lie \( n \) lines of the congruence.

A bilinear congruence \( [|d_1|, |d_2|] \sim \) a congruence \( (4n, 2n) \). The transformation \( (1) \) is involutorial. Hence the conjugate congruence of the \( (1, 1) \) will be of order equal to the number of lines common to the \( (1, 1) \) and the conjugate of an arbitrary bundle, which is \( 4n \). Likewise, the class will be the number of lines common to the \( (1, 1) \) and the conjugate of an arbitrary plane field.

A linear complex is transformed by \( (1) \) into a complex of order \( (n+1) \) since this is the order of the transformation.

3. Homology-Geiser. Consider \( I_\alpha \) in \( \alpha \) as before, and in \( \beta \), \( I_\beta: P^\beta \cdot \cdot \cdot P^\beta \), with invariant curve \( \Delta_\beta \) of order 6 having double-points at \( P_1, \cdot \cdot \cdot, P_7 \). The order of the transformation is 9; the invariant lines form a congruence \( (6, 6) \) and a cone of order 6, vertex \( O_1 \), base curve \( \Delta_\beta \); the singular elements are, as
before, the lines of the bundles whose vertices are $F$-points of $I_\beta$, a regulus $\{c\}_9$ of order 9 and the plane fields $(\alpha), (\beta)$. To any line of any bundle corresponds a cone of order 3; to any generator of $\{c\}_9$, a bundle, vertex on $c$; to $(\alpha)$ a special complex of order 8, and to $(\beta)$ a special linear complex. If $t_\beta$ pass through one or two points $P_i$, its conjugate cone is of order 5 or 2. If the vertex $T$ of a pencil $(T, \beta)$ be at $P_i$, the conjugate congruence of the pencil is $(6, 5)$; otherwise $(T, \alpha)$ or $(T, \beta) \sim_a$ a congruence $(9, 8)$.

If the plane $\tau$ of an arbitrary pencil $(T, \tau)$ pass through one or two points $P_i$, the conjugate regulus is of order 6 or 3. Otherwise $(T, \tau) \sim_a$ a regulus of order 9.

If the plane $\tau$ pass through $j$ points $P_i(j=0, 1, 2)$, $(\tau) \sim_a$ a congruence $(8 - 3j, 8 - 3j)$.

An arbitrary bundle $(T) \sim_a$ a congruence $(24, 8)$.

A congruence $(1, 1) \sim_a$ a congruence $(32, 16)$, and a linear complex $\sim_a$ a complex of order 9.

4. Homology-Bertini. Consider $I_a$ as before and $I_\beta: P_1^\delta \cdots P_8^\delta$, $\Delta_\beta$ being of order 9 with triple-points at $P_i$. The order of the transformation is 18; the invariant lines form a congruence $(9, 9)$ and a cone of order 9; the singular elements are the lines of the bundles $(P_i)$, a regulus $\{c\}_1^{18}$ of order 18, and the plane fields $(\alpha), (\beta)$. Any line of any $(P_i) \sim_a$ a cone of order 6, each generator of $\{c\}_1^{18} \sim_a$ a bundle with vertex on $c$, $(\alpha) \sim_a$ a special complex of order 17, and $(\beta) \sim_a$ a special linear complex. The conjugate cone of a line $t_\beta$ has its order $17 - 6j$, where $j$ is the number of points $P_i$ on $t_\beta$. The conjugate congruence of a pencil $(T, \alpha)$ or $(T, \beta)$ is $(18, 17)$ unless $T$ lie at some $P_i$, when the congruence is $(12, 11)$.

If $\tau$ pass through $j$ points $P_i(j=0, 1, 2)$, then the pencil $(T, \tau) \sim_a$ a regulus of order $18 - 6j$ and the plane field $(\tau) \sim_a$ a congruence $(17 - 6j, 17 - 6j)$.

A bundle $(T) \sim_a$ a congruence $(51, 17)$.

A congruence $(1, 1) \sim_a$ a congruence $(68, 34)$, and the transform of a linear complex is a complex of order 18.

5. Jonquières-Jonquières. When we consider two perspective Jonquières involutions, $I_a$ of order $m$, center $O_1$, and $I_\beta$ of order $n$, center $P_1$, the order of the transformation is $(m + n)$. The invariant lines form a congruence $(mn, mn)$ since now $\Delta_a$ and $\Delta_\beta$
are of orders \( m, n \), respectively, and each passes simply through the simple \( F \)-points in its respective plane and multiply through the center. In addition to the singular regulus \( \{ c \}^{m+n} \) of order \( m+n \) whose generators are transformed into bundles with vertices on \( c \), and the singular plane fields \( (\omega), (\beta) \) whose conjugates are special complexes of orders \( n, m \), respectively, and the singular bundles whose vertices are at \( F \)-points and each of whose lines is transformed into a cone whose order is the multiplicity of the \( F \)-point in \( I_{\alpha} \) or \( I_{\beta} \), there are \((2m-1)(2n-1)\) singular lines \( O_i P_j \) whose conjugates are congruences. If \( i, j \neq 1 \), each congruence is \((1, 1)\); if \( i=1, j \neq 1 \), each congruence is \((m, m)\); if \( i \neq 1, j = 1 \), each is \((n, n)\); if \( i = j = 1 \), the conjugate congruence is \([m-1][n-1], [m-1][n-1]\). A line \( (y) \equiv t_{\alpha} \sim \) a cone of order \((n-j)\), where \( j \) is the sum of the multiplicities of \( F \)-points \( O_i \) on \( t_{\alpha} \), and \( (y) \equiv t_{\beta} \sim \) a cone of order \((m-i)\), where \( i \) is the sum of the multiplicities of \( F \)-points \( P_i \) on \( t_{\beta} \). A pencil \( (T, \beta) \sim \) a congruence \((m+n-i, m[n-i])\) where \( i \) is the multiplicity of \( T \) as an \( F \)-point of \( I_{\beta} \) and \( (T, \alpha) \sim \) a congruence \((m-j+n, [m-j]n)\), where \( j \) is the multiplicity of \( T \) as an \( F \)-point of \( I_{\alpha} \).

An arbitrary pencil \( (T, \tau) \sim \) a regulus \( R \) of order \((m-i+n-j)\), where \( i \) is the sum of the multiplicities of \( F \)-points \( O_i \) and \( j \) the sum of the multiplicities of \( F \)-points \( P_i \) lying in \( \tau \). If \( i = m \) or \( j = n \), then \( R \) is a cone. Both \( i = m \) and \( j = n \) will not occur if \( I_{\alpha}, I_{\beta} \) be taken arbitrarily.

A plane field \( (\tau) \sim \) a congruence whose order and class are both \((m-i)(n-j)\), where \( i, j \) are as defined immediately above. In the event \( m-i=0 \) or \( n-j=0 \), the conjugate of \( (\tau) \) is a cone whose order is the factor which does not vanish.

An arbitrary bundle \( (T) \sim \) a congruence \((3mn, mn)\).

A congruence \((1, 1) \sim \) a congruence \((4mn, 2mn)\), and a linear complex has for conjugate a complex of order \((m+n)\).

6. Jonquières-Geiser. Take \( I_{\alpha} \) as in §5 and \( I_{\beta} \) as in §3. The transformation is of order \((m+8)\); the invariant lines form a congruence \((6m, 6m)\); the singular elements are the bundles \((O_i), (P_i)\), the \(7(2m-1)\) lines \( O_i P_i \), the plane fields \( (\alpha), (\beta) \), and the singular regulus \( \{ c \}^{m+8} \) defined as in each previous case. Each line of \( (O_i) \sim \) a cone of order \((m-k)\), where \( k \) is the multiplicity of \( O_i \) in \( I_{\alpha} \), and each line of \( (P_i) \sim \) a cone of order 3;
(y) = O_i P_j \sim \text{a congruence whose order and class are both } 5(m-k), \text{where } k \text{ is as defined just above}; (\alpha) \sim \text{a special complex of order } 8 \text{ and } (\beta) \sim \text{a special complex of order } m; \text{each generator of } \{c\}^{m+8} \sim \text{a bundle whose vertex is on } c. \text{Finally, } (y) = t_\alpha \sim \text{a cone of order } (m-i), \text{where } i \text{ is the sum of the multiplicities of } F\text{-points on } t_\alpha, \text{and } (y) = t_\beta \sim \text{a cone of order } (8-j), \text{where } j \text{ is the sum of the multiplicities of } F\text{-points on } t_\beta.

A pencil \((T, \alpha)\) \sim \text{a congruence } (m-i+8, [m-i]8), \text{where } i \text{ is the multiplicity of } T \text{ as an } F\text{-point of } I_\alpha, \text{and a pencil } (T, \beta) \sim \text{a congruence } (m+8-j, m[8-j]), \text{where } j \text{ is the multiplicity of } T \text{ as an } F\text{-point of } I_\beta.

An arbitrary pencil \((T, \tau)\) \sim \text{a regulus } R \text{ whose order is } (m-i+8-j), \text{where } i \text{ is the sum of multiplicities of } F\text{-points } O_k \text{ and } j \text{ the sum of multiplicities of } F\text{-points } P_l \text{ lying in } \tau. \text{If } I_\alpha, I_\beta \text{ be taken arbitrarily, no more than } 3 \text{ points } O_k, P_l \text{ lie on } \tau. \text{If } i=m, R \text{ is a cone of order } (8-j).

A plane field \((\tau)\) \sim \text{a congruence whose order and class are both } (m-i)(8-j), \text{where } i, j \text{ are as defined in the preceding paragraph. If } i=m, \text{the conjugate of } (\tau) \text{ is a cone of order } (8-j).

A bundle \((T)\) \sim \text{a congruence } (24m, 8m).

A congruence \((1, 1)\) \sim \text{a congruence } (32m, 16m) \text{ and a linear complex is transformed into complex of order } (m+8).

7. Jonquières-Bertini. Take \(I_\alpha \) as in §5 and \(I_\beta \) as in §4. The transformation is of order \((m+17)\), and all of the results follow by replacing 8 by 17 in the preceding section, except that a bundle \((T)\) \sim \text{a congruence } (51m, 17m) \text{ and a congruence } (1, 1) \sim \text{a congruence } (68m, 34m).

8. Geiser-Geiser. Given \(I_\alpha: O_i^8 \cdots O_i^8 \) \text{ and } \(I_\beta \) as in §3. The transformation is of order 16; the invariant lines form a congruence \((36, 36)\); the singular elements are \((O_i), (P_i), O_i P_j, \{c\}^{16}, (\alpha), (\beta), \) \text{and their conjugates are easily found by the methods outlined above. The conjugates of the elementary forms } (T, \tau), (\tau), (T), (1, 1) \text{ and linear complex are also readily obtained.}

9. Geiser-Bertini. Consider \(I_\alpha \) as in §8 and \(I_\beta \) as in §4. The transformation is of order 25; the invariant lines form a congruence \((54, 54)\); the singular elements are \((O_i), (P_i), O_i P_j, \{c\}^{35}, (\alpha), (\beta), \) \text{and their conjugates, and those of the elemen-}
tary forms follow immediately from the method we have used throughout this paper.

10. Bertini-Bertini. Given $I_a: O_0 \cdots O_6$ and $I_b$ as in §4. The order of the transformation is 34, and by repetition of what has been done above we can discuss this transformation completely.

If fundamental elements of one or both Cremona plane involutions lie on the line $c = \alpha \beta$ the preceding results must be modified. The details are not difficult in each particular case, but the large number of possible forms cannot be considered here.

CORNELL UNIVERSITY

A SEPARATION THEOREM*

BY W. A. WILSON

Various writers on topology have had occasion in the course of their work to prove lemmas of the following general nature. Given sets $A$ and $B$ lying in a connected space $Z$, it is possible to express $Z$ as the union of two continua $M$ and $N$ such that $N \cdot (A - A \cdot B) = M \cdot (B - A \cdot B) = 0$, provided that $A$, $B$, and $Z$ satisfy the proper conditions. The last of these to come to the writer's attention are two theorems by Vietoris and one by the author of this note.† Such theorems are of course generalizations of Tietze's separation axioms‡ and it might prove profitable to work out systematically the possibilities along this line.

In some of the generalizations mentioned it is shown that, if $Z$ is locally connected, a decomposition $Z = M + N$, where $M$ and $N$ are also locally connected, is possible, but, as far as the writer knows, the following theorem, which shows a certain kind of local connectivity for $M \cdot N$ as well as for $M$ and $N$, is new.

**Theorem.** Let $A$ and $B$ be sub-continua of the locally connected compact metric space $Z$ and let $A \cdot B$ be totally disconnected or void.

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