A NOTE ON THE DICKSON THEOREM ON UNIVERSAL TERNARIES*

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1. Introduction. A form $f$ with integer coefficients in integer variables is called universal if it represents all positive and negative integers. Evidently, since $f$ is homogeneous, it represents zero for the variables all zero. In case $f = 0$ for integral values of the variables not all zero $f$ is called a zero form.

L. E. Dickson† has given a number-theoretic proof of his theorem that every universal ternary quadratic form is a zero form. But his proof is highly technical and consequently quite long and complicated. In the present note I shall give an almost trivial rational proof of Dickson's result. I shall also prove a generalization of his theorem for ternaries over any non-modular field $F$.

2. Quadratic Forms over $F$. Let $F$ be any non-modular field and let $f(x_1, \cdots, x_n)$ be an $n$-ary quadratic form over $F$. Then we shall call $f$ a zero form if $f = 0$ for $x_1, \cdots, x_n$ in $F$ and not all zero. We shall also say that, if every $\rho$ in $F$ is represented by $f$ for $x_1, \cdots, x_n$ in $F$, the form $f$ is universal over $F$.

It is well known‡ that there exists a non-singular linear transformation $x_i = \sum a_{ij}x_j$ with $a_{ij}$ in $F$ such that

$$f(x_1, \cdots, x_n) = \phi(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{r} g_i x_i^2 + 0 \cdot \sum_{j=r+1}^{n} x_j^2,$$

with $g_i \neq 0$ in $F$. The integer $r$ is the rank of $f$. Evidently $f$ is a zero form if and only if $\phi$ is a zero form. But if $r < n$, the form $\phi$ vanishes for any $X_n$ in $F$, if $X_1 = \cdots = X_r = 0$.

Theorem 1. Every $n$-ary of rank $r < n$ is a zero form. Every $n$-ary of rank $n$ is equivalent to

$$g_1X_1^2 + g_2X_2^2 + \cdots + g_nX_n^2,$$

with $g_i$ all not zero.

* Presented to the Society, April 15, 1933.
‡ See Dickson, Modern Algebraic Theories, p. 69
3. Proof of the Dickson Theorem. Let \( f(x, y, z) \) be a universal ternary. By Theorem 1 either \( f \) is a zero form of rank less than three or

\[
f(x, y, z) = \phi(X, Y, Z) = \alpha X^2 + \beta Y^2 - \gamma Z^2,
\]

where \( \alpha, \beta, \gamma \) are rational, \( \alpha \beta \gamma \neq 0 \), and \( X, Y, Z \) are linearly independent rational linear functions of \( x, y, z \). Define

\[
\delta = \gamma (\alpha \beta)^{-1}, \quad a = \alpha \delta, \quad b = \beta \delta, \quad ab = - (\alpha \beta \delta) \delta = - \gamma \delta,
\]

so that, for a rational number \( \delta \neq 0 \),

\[
\delta f = \delta \phi = \psi(X, Y, Z) = a X^2 + b Y^2 - ab Z^2.
\]

Write \( \delta = \delta_1 \delta_2^{-1} \), where \( \delta_1 \) and \( \delta_2 \) are integers. Since \( f \) is universal, \( f(x, y, z) = \delta_1 \delta_2 \) for integer \( x, y, z \). Then if \( x_0 = x \delta_1^{-1}, y_0 = y \delta_1^{-1}, z_0 = z \delta_2^{-1} \), we have \( f(x_0, y_0, z_0) = \delta_1^{-2} \delta_1 \delta_2 = \delta_1 \delta_2^{-2} \delta_1^{-1} = \delta^{-1} \), for rational \( x_0, y_0, z_0 \). Hence we have proved the following fact.

**Lemma 1.** If \( f \) is universal, \( \phi = \delta^{-1} \) for rational \( X, Y, Z \).

Let then \( \delta^{-1} = \phi, \quad \psi = \delta \phi = \delta \delta^{-1} = 1 = a X^2 + b Y^2 - ab Z^2 \), and write as a consequence

\[
\delta f = \delta \phi = \psi(X, Y, Z) = a X^2 + b Y^2 - ab Z^2.
\]

If \( \xi = 0 \), put \( \eta = 1, \xi = X \), so that

\[
\psi(\xi, \eta, \xi) = b \cdot 1^2 - ab \cdot X^2 = b(1 - a X^2) = b \xi = 0
\]

for \( \eta \neq 0 \), and \( \phi = \delta^{-1} \psi \) is a zero form. Hence \( f \) is a zero form, since \( f = 0 \) for rational \( x, y, z \) not all zero if and only if \( f = 0 \) for integers \( x, y, z \), not all zero, since \( f \) is homogeneous.

Let then \( \xi \neq 0 \), and put \( \eta = a(Z - X Y), \xi = Y - a X Z, \) so that

\[
\begin{align*}
b \eta^2 - ab \xi^2 &= b [a^2(Z^2 - 2 XYZ + X^2 Y^2) \\
&- a(Y^2 - 2 a XYZ + a^2 X^2 Z^2)] \\
&= - ab [(Y^2(1 - a X^2) - a Z^2(1 - a X^2))] \\
&= - a(1 - a X^2)(b Y^2 - ab Z^2) = - a \xi^2,
\end{align*}
\]

\( \delta \phi(\xi, \eta, \xi) \equiv a \xi^2 + b \eta^2 - ab \xi^2 = 0 \),

and \( \phi(\xi, \eta, \xi) = 0 \) for \( \xi \neq 0 \). Hence again \( \phi \), and therefore also \( f \), are zero forms, and we have proved the Dickson Theorem. The above proof is a rational proof holding for any field \( F \) so we have immediately the following result.
Lemma 2. If a ternary \( f(x, y, z) \) with coefficients in \( F \) represents the associated quantity \( \delta^{-1} \), then \( f \) is a zero form.

4. Universal Ternaries over \( F \). We shall now prove the following theorem.

Theorem 2. A non-singular ternary quadratic form over \( F \) is universal over \( F \) if and only if it is a zero form.

For let \( f \) be a zero form, so that \( f(x, y, z) = 0 \) for \( x, y, z \) not all zero and in \( F \). Then

\[
\delta \phi = \psi(\xi, \eta, \zeta) = a\xi^2 + b\eta^2 - ab\zeta^2 = 0
\]

for \( \xi, \eta, \zeta \) not all zero and in \( F \). Let \( \rho \) be any quantity of \( F \), \( \sigma = \rho \delta \). If \( \xi = 0 \), then \( b(\eta^2 - a\zeta^2) = 0 \), whence \( \eta^2 = a\zeta^2 \), so that \( \zeta \neq 0 \). Thus write \( \xi_0 = \zeta \eta^{-1} \), from which \( a\xi_0^2 = 1 \). Put

\[
X = 0, \quad Y = \frac{\sigma + b^{-1}}{2}, \quad Z = \frac{\sigma - b^{-1}}{2} \xi_0,
\]

so that, since \( 1 = a\xi_0^2 \),

\[
4\psi(X, Y, Z) = b[(\sigma + b^{-1})^2 - (\sigma - b^{-1})^2a\xi_0^2]
\]

\[
= b[(\sigma + b^{-1})^2 - (\sigma - b^{-1})^2]
\]

\[
= 4bb^{-1}\sigma = 4\sigma, \quad \text{and} \quad \psi = \sigma.
\]

Then \( \phi = \delta^{-1}\sigma = \rho \) and hence \( f = \rho \) for corresponding \( x, y, z \) in \( F \).

Next let \( \xi \neq 0 \). Then \( a + b(\eta\xi^{-1})^2 - ab(\zeta\xi^{-1})^2 = 0 \), and if we write \( \eta\xi^{-1} = a\xi_0, \zeta\xi^{-1} = \eta_0 \), we have \( a + a^2b\zeta^2 - ab\eta_0 = 0 \), \( 1 = b\eta^2 \), \(-ab\zeta^2 \). Then put

\[
X = \frac{\sigma + a^{-1}}{2}, \quad Y = \frac{\sigma - a^{-1}}{2}a\xi_0, \quad Z = \frac{\sigma - a^{-1}}{2} \eta_0,
\]

whence

\[
4\psi(X, Y, Z) = a(\sigma + a^{-1})^2 + (ba^2\zeta^2 - ab\eta_0)(\sigma - a^{-1})^2
\]

\[
= a[\sigma + a^{-1})^2 - (\sigma - a^{-1})^2] = 4aa^{-1}\sigma = 4\sigma,
\]

\[
\psi = \sigma, \quad \phi = \delta^{-1}\sigma = \rho.
\]

Hence in this case also \( f = \rho \) as desired, so that \( f \) is universal.
Conversely let \( f \) be universal. Then \( f \) represents \( \delta^{-1} \) and, by Lemma 2, is a zero form. This proves* Theorem 2.

It is well known† that the determinant of the form \( \phi(X, Y, Z) \) equivalent to \( f \) is \( h^2d \), where \( h \) is the determinant of the transformation. Hence \( -\alpha\beta\gamma = h^2d \), so that

\[
\delta = \gamma(\alpha\beta)^{-1} = (\alpha\beta\gamma)(\alpha\beta)^{-2} = -dh^2(\alpha\beta)^{-2} = -dk^2,
\]

where \( k \) is in \( F \). Then

\[
-df = dk^{-2}\phi = k^{-2}\psi(X, Y, Z) = \psi(\xi, \eta, \zeta),
\]

for \( X = k\xi \cdot X = k\eta \cdot Z = k\zeta \). Hence if \( f \) represents the negative of its determinant, the form \( \psi = -df = (-d)^2 \) represents \( d^2 \), and hence unity, and hence \( f \) is a zero form by Lemma 2. We may therefore replace Lemma 2 by the following statement.

**Theorem 3.** If \( f \) is a ternary with non-zero determinant \( d \), then

\[-df(x, y, z) = \psi(X, Y, Z) = aX^2 + bY^2 - abZ^2 \]

for a suitable transformation. Also \( f \) is a universal zero form if and only if \( f \) represents \(-d\).

In particular the above Theorem 2 holds for the case where \( F = \mathbb{R} \), the field of all rational numbers. If, however, \( a \) is any rational number, then \( a = b^{-2}c \), where \( b \) and \( c \) are integers. Obviously, if \( f = a \) for rational \( x, y, z \), then \( f = c \) for rational \( x, y, z \). Hence we have proved a partial converse to Dickson’s theorem.

**Theorem 4.** A non-singular ternary quadratic form with integer coefficients is a zero form if and only if it represents all integers for rational values of its variables.

* It is evident that Theorem 2 is true if it can be proved for forms of type of \( \psi(X, Y, Z) = aX^2 + bY^2 - abZ^2 \). If \((1, i, j, ij)\), \(i^2 = a, j = b, ji = -ij\), is a generalized quaternion algebra over \( F \), then for \( ab \neq 0 \), this algebra is either a division algebra or a total matric algebra. If \( g = Xi + Yj + Zij \), then \( g^2 = \psi(X, Y, Z) \). Hence, if \( \psi \) is a zero form, the algebra \( Q \) is not a division algebra and there exists a two-rowed matrix whose square is \( \sigma \) so that \( \psi \) represents \( \sigma \). The converse of Theorem 2 is similarly proved. It is in fact this linear algebra theorem (which has long been known to me) which gave me an immediate proof of Theorem 2 as soon as I discovered the reduction given by (1)–(3).

† See Dickson, *Modern Algebraic Theories*, pp. 64–70.