NORMAL DIVISION ALGEBRAS OVER ALGEBRAIC NUMBER.Fields NOT OF FINITE DEGREE*

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1. Introduction. If $R$ is the field of all rational numbers and if $\xi_1, \cdots, \xi_n$ are ordinary algebraic numbers, then the field $\Omega = R(\xi_1, \cdots, \xi_n)$ of all rational functions with rational coefficients of $\xi_1, \cdots, \xi_n$ is an algebraic number field of finite degree (the maximum number of linearly independent quantities of $\Omega$) over $R$. It has recently been proved† that every normal simple algebra over such a field $\Omega$ is cyclic. In particular it has been shown that every normal division algebra of order $n^2$ (degree $n$) over $\Omega$ is cyclic and has exponent $n$.

In the present note I shall give an extension of the above results to normal division algebras over any algebraic number field $\Lambda$. I shall prove that all normal division algebras over $\Lambda$ are cyclic and with degree equal to exponent but shall give a trivial example showing that the theorem corresponding to the above on normal simple algebras is false. The problem of the equivalence of normal division algebras over $\Lambda$ will also be discussed.

2. Cyclic Algebras. Let $F$ be any non-modular field and let $Z$ be cyclic of degree $n$ over $F$. Then $Z$ possesses a generating automorphism

$$S: z \mapsto z^S, \quad (z \in Z, z^S \in Z),$$

such that every automorphism of $Z$ is one of $S^0 = S^n = I, S, S^2, \cdots, S^{n-1}$. The algebra $A$ of all quantities

$$\sum_{i=0}^{n-1} z_i y^i, \quad (z \in Z),$$

is a cyclic algebra with multiplication table

$$y^n = \gamma \text{ in } F, \quad y^e z = z^{S^e} y^e, \quad (e = 0, 1, \cdots),$$

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for every $z$ of $Z$. Evidently $A$ is uniquely defined by $Z, S, \gamma$, and thus we write

$$A = (Z, S, \gamma).$$

Let $F$ be contained in any larger field $K$. Then

$$A_K = (Z, S, \gamma)_{K}$$

is the algebra with the same basis and constants of multiplication as $A$, but over $K$.

If $A_K$ is a division algebra, then so evidently is $A$. But then $Z_K$, which is the algebra with the same basis and constants of multiplication as the field $Z$, but over $K$, is a field and in fact is evidently cyclic of degree $n$ over $K$. Evidently $A = (Z_K, S, \gamma)$ over $K$.

**Theorem 1.** Let $A = (Z, S, \gamma)$ over $F$, $F < K$, and let $A_K$ be a division algebra. Then $A_K$ is the cyclic algebra $(Z_K, S, \gamma)$ over $K$.

3. The Determination of Algebras over $\Lambda$. Let $\Lambda$ be any non-modular field whose quantities are all algebraic numbers and let $A$ be a normal division algebra of order $m = n^2$ over $\Lambda$. If $u_1, \cdots, u_m$ are a basis of $A$, then $u_i u_j = \sum \gamma_{ijk} u_k$ with $\gamma_{ijk}$ in $\Lambda$. But then $\gamma_{ijk}$ are all algebraic numbers, so that $L = \mathcal{R}(\gamma_{111}, \cdots, \gamma_{ijk}, \cdots, \gamma_{mnm})$ is algebraic of finite degree.

The linear set $B = (u_1, \cdots, u_m)$ over $L$ is evidently an algebra of order $m$ over $L$. If in particular $u_1 = 1$, the modulus of $A$, then $u_1$ is the modulus of $B$. Evidently $A = B_\Lambda$.

If $B$ contains any divisors of zero, then these quantities are in the division algebra $A$, a contradiction. Hence $B$ is a division algebra.

Let $B$ contain a quantity $k = \sum \lambda_i u_i$, $\lambda_i$ in $L$, which is commutative with every quantity of $B$. In particular $k u_i = u_i k$, so that $k (\sum \mu_i u_i) = (\sum \mu_i u_i) k$ for $\mu_i$ any quantities of the field $\Lambda$. But $A$ is normal, so that $k$ is a multiple of the modulus $u_1$ of $A$ by a quantity of $\Lambda$. Hence $k = \mu u_1 = \sum \lambda_i u_i$. Since the $u_i$ are linearly independent in $\Lambda$, we have $\mu = \lambda_1$, $k$ is a multiple of $u_1$ by a quantity of $L$, and $B$ is normal.

The normal division algebra $B$ of degree $n$ over $L$ is thus* a cyclic algebra $(Z, S, \gamma)$ over $L$. The basis, $(u_i)$, of $A$ is linearly independent in $L$.

* By the result already quoted on normal division algebras over $\Omega$. 
expressible with coefficients in $L$ in terms of the basis of $B = (Z, S, \gamma)$ in its cyclic form, so that in fact $A = (Z, S, \gamma)_\Lambda$. By Theorem 1 we have the following result.

**Theorem 2.** Let $A$ be a normal division algebra of degree $n$ over an algebraic number field $\Lambda$ not of finite degree. Then there exists a sub-field $L$ (of $\Lambda$) of finite degree and a cyclic algebra, $B = (Z, S, \gamma)$, over $L$ such that $A = (Z_\Lambda, S, \gamma)_\Lambda$ over $\Lambda$, where $Z_\Lambda$ is a cyclic field of degree $n$ over $\Lambda$. Hence $A$ is cyclic.

**4. The Exponent of Algebras $A$.** Suppose that the algebra $A$ of Theorem 2 has exponent $\rho < n$. Then $A^\rho$ is well known to be equal to $M^{\rho-1} \times (Z, S, \gamma^\rho)$, where $M$ is a total matric algebra. But $A^\rho$ is a total matric algebra; hence $(Z_\Lambda, S, \gamma^\rho)$ is also. Hence $\gamma^\rho$ is the norm $N(c)$ of a quantity $c$ of $Z_\Lambda$.

Let $Z = L(x), Z_\Lambda = \Lambda(x)$, so that $c = \sum c_i x_i$, where the $c_i$ are in $\Lambda$. The field $L = L(c_0, \cdots, c_{n-1})$ is algebraic of finite degree. Moreover, if $B = (Z, S, \gamma)$, then evidently $Z_0 = L_0(x)$, $B_0 = (Z_0, S, \gamma)$ over $L_0$, is contained in $A$ and hence is a cyclic division algebra. But $B^\rho_0 = (Z_0, S, \gamma^\rho) \times M^{\rho-1}$ is a total matric algebra, since $\gamma^\rho = N(c)$, where $c$ is in $Z_0$.

The exponent of $B_0$ of degree $n$ over $L_0$ is known to be $n$ since $B$ is a cyclic division algebra over $L_0$, which is algebraic of finite degree. Hence $\rho \geq n$, a contradiction.

**Theorem 3.** The exponent of any normal division algebra over $\Lambda$ is its degree.

**5. On the Equivalence of Algebras over $\Lambda$.** Let $A = (Z_\Lambda, S, \gamma)$ and $C = (Y, T, \delta)$ over $\Lambda$ be normal division algebras. Then $Z$ and $\gamma$ are obtained with respect to a field $L_1$ defined by $A$, $Y$, and $\delta$ with respect to $L_2$ defined by $C$. If $L$ is the composite of $L_1$ and $L_2$, then we may evidently take $L$ as the common field of Theorem 2 for both algebras $A$ and $C$. Hence $A = (Z, S, \gamma)_\Lambda, (Z, S, \gamma)$ a normal division algebra over $L, C = (Y, T, \delta)_\Lambda, (Y, T, \delta)$ also a normal division algebra over $L$.

The algebra $A$ is equivalent to the algebra $C$ if and only if $A \times C^{-1} = (Z, S, \gamma) \times (Y, T, \delta^{-1})$ is a total matric algebra. But, as is well known, $(Z, S, \gamma) \times (Y, T, \delta^{-1}) = (X, R, \epsilon) \times M$, where $M$ is a total matric algebra and $(X, R, \epsilon)$ is a uniquely determined cyclic algebra. Evidently $A \times C^{-1}$ is total matric if and only if $(X, R, \epsilon)_\Lambda$ is total matric. For $A \times C^{-1} = M \times (X, R, \epsilon)_\Lambda$.,
But then $e = N(c)$, where $c$ is in $X_\Lambda$. As before there exists a sub-field $L_0$ of $\Lambda$ of finite degree such that $c$ is in $X_{L_0}$, $(X, R, \epsilon)_{L_0}$ is total metric. But then $(Z, S, \gamma)_{L_0}$ is equivalent to $(Y, T, \delta)_{L_0}$. The converse is obvious and we have proved this theorem.

**Theorem 4.** Let $A$ and $C$ be normal division algebras of degree $n$ over $\Lambda$, an algebraic field not of finite degree, so that $A = (Z_\Lambda, S, \gamma)$, $C = (Y_\Lambda, T, \delta)$, where $B = (Z, S, \gamma)$, $D = (Y, T, \delta)$ are cyclic over the same sub-field $L$ of finite degree of $\Lambda$. Then $A$ and $C$ are equivalent if and only if there exists a sub-field $L_0 > L$ of $\Lambda$ such that $L_0$ has finite degree and the algebras $B_{L_0}$ and $D_{L_0}$ are equivalent.

The above theorem essentially reduces the problem of the equivalence of normal division algebras over $A$ to the corresponding problem (already solved*) for algebras over fields of finite degree, and to a consideration of the sub-fields of $\Lambda$ of finite degree.

6. **Normal Simple Algebras over $\Lambda$.** In this section we shall show trivially that there exist non-cyclic normal simple algebras over an algebraic field $\Lambda$. We take $\Lambda$ to be the field of all constructible (with ruler and compass) numbers, extended by $i = (-1)^{1/2}$. That is, we take $\Lambda$ to consist of all numbers obtained from rational numbers by a finite number of rational operations and extractions of square roots.

Evidently any equation $x^2 = c$, $c$ in $\Lambda$, is reducible in $\Lambda$ since $c^{1/2}$ is also in $\Lambda$. But then there exist no cyclic algebras of degree two over $\Lambda$. Hence the total metric algebra of degree two over $\Lambda$, a normal simple algebra, is non-cyclic.

* See Hasse, loc. cit.