ON THE NUMBER OF APPARENT DOUBLE POINTS ON A CERTAIN $V^a_s$ IN AN $S_{2k+1}$

BY B. C. WONG

Consider a $k$-dimensional variety, $V^a_s$, of order $n$ which is the locus of a single infinity of $(k-1)$-spaces in an $S_{2k+1}$. It is known that such a $V^a_s$, if it is rational, that is, if its section by a general $S_{k+2}$ of $S_{2k+1}$ is a rational curve, has

$$b_k = \frac{1}{2}(n - k)(n - k - 1)$$

apparent double points.† The question arises: What is the value of $b_k$ when $V^a_s$ is not rational? The case $k = 1$ is familiar; a curve of order $n$ in an $S_3$ has

$$b_1 = \frac{1}{2}(n - 1)(n - 2) - p$$

apparent double points, where $p$ is the deficiency of the curve. It is also known that, for $k = 2$, the number of apparent double points on a ruled surface $F^n$ of order $n$ in an $S_5$ is‡

$$b_2 = \frac{1}{2}(n - 2)(n - 3) - 3p,$$

where $p$ is the deficiency of the curve of intersection of $F^n$ by a general $S_4$ of $S_5$. For $k > 2$, the number $b_k$ of apparent double points of a $V^a_s$ in an $S_{2k+1}$ seems to be as yet unknown. It is our purpose in this note to derive a formula for this number.

Now let $V^a_s$ be intersected by a general $S_{k+2}$ of $S_{2k+1}$ in a curve $C^n$ of deficiency $p$. If $p > 0$, we say that $V^a_s$ is not rational. We shall say that $p$ is also the deficiency of $V^a_s$ and shall regard $n$ and $p$ as the two essential characteristics of the variety as all its other characteristics can be expressed in terms of them for a

* B. C. Wong, On the number of $(q+1)$-secant $S_{q+1}$'s of a certain $V^n_s$ in an $S_{2k+2}^{k+1}$, this Bulletin, vol. 39, pp. 392–394.

† By an apparent double point of a $V^a_s$ we mean a secant line of $V^a_s$ passing through a given point of $S_{2k+1}$. The projection in an $S_{2k}$ of $V^a_s$ will have $b_k$ improper double points each of which is the projection of an apparent double point of $V^a_s$.

given value of $k$. Then the formula for $b_k$ must be a function of $n$ and $p$, and of $k$ also.

Consider the ruled surface $F^n$ in which $V_k^n$ is met by a general $S_{k+3}$ of $S_{2k+1}$. The projection of $F^n$, if it is in an $S_5$, has $b_3$, given by formula (2), apparent double points; and, if it is in an $S_3$, has a double curve of order $b_1$ given by formula (1). On this double curve lie a finite number, $j_1$, of pinch points. This number is known and will be given subsequently.

Next, consider the planed variety $V_k^n$ common to $V_k^n$ and a general $S_{k+4}$ of $S_{2k+1}$. The projection in an $S_7$ of this $V_k^n$ has $b_3$ apparent double points. Projecting this projection successively upon an $S_9$, an $S_8$, and an $S_6$, we see that the resulting variety in $S_6$ has $b_3$ improper double points; that the one in $S_8$ contains a double curve of order $b_2$ upon which lie $j_2$ pinch points; and, finally, that the one in $S_4$ contains a double surface of order $b_1$ upon which lies a pinch curve of order $j_1$.

In general, an $S_{k+h+1}$ of $S_{2k+1}$ meets $V_k^n$ in a $V_k^n$ which is the locus of a single infinity of $(h-1)$-spaces. Now if we let $V_k^n$ be projected upon an $S_{2k-i}$, $(i = 0, 1, \ldots, h-1)$, of $S_{k+h+1}$, then we have for projection an $h$-dimensional variety of order $n$ with a double $i$-dimensional variety of order $b_{k-i}$ and an $(i-1)$-dimensional pinch variety of order $j_{k-i}$ lying on the double variety. If $i = 0$, the projection in $S_{2h}$ has $b_h$ improper double points.

Suppose $h = k$, and then we have the given $V_k^n$ itself. A general $S_{2k-i}$-projection of this $V_k^n$ contains a double $i$-dimensional variety of order $b_{k-i}$ upon which lies an $(i-1)$-dimensional variety of order $j_{k-i}$.

In order to determine $b_k$ we find it necessary to determine $b_h$. This determination will be much facilitated if we make use of the two following results already known.

(A) The characteristics $b_0, b_1, \ldots, b_h; j_0, j_1, \ldots, j_{k-1}$ of any $V_k^n$ in $r$-spaces satisfy the relations*

$$2b_h = 2b_{k-1} - j_{k-1}$$

$$= 2b_{k-2} - j_{k-1} - j_{k-2}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

* B. C. Wong, *On certain characteristics of $k$-dimensional varieties in $r$-space*, this Bulletin, vol. 38, pp. 725-730. The notations used in this paper are slightly different from those adopted in the present work.
1933-1

APPARENT DOUBLE POINTS 757

\[ b_1 = j_{h-1} - j_{h-2} - \cdots - j_1 \]
\[ b_0 = j_{h-1} - j_{h-2} - \cdots - j_1 - j_0. \]

Here \( b_0 \) is to be taken equal to \( n(n-1)/2 \) and \( j_0 \) is the rank of the curve \( C^a \) common to \( V_k^n \) and an \( S_{k+2}. \)

(B) The number of pinch points on the double curve of a \( V_h^n \) which is the locus of a single infinity of \( S_{h-1}'s \) in an \( S_{2h-1} \) is

\[ j_{h-1} = 2(n - h + h^2). \]

Combining these two results, we find that

\[ b_h = b_0 - (1/2) \sum_{i=0}^{h-1} j_i = (n - h)(n - h - 1)/2 - k(1 + 1)p/2. \]

If \( h = 1 \) and \( 2 \), we have formulas (1) and (2), respectively. For \( h = k \), we have

\[ b_k = (n - k)(n - k - 1)/2 - k(k + 1)p/2 \]

as the number of apparent double points on a \( V_h^n \) which is the locus of a single infinity of \( S_{k-1}'s \) in an \( S_{2k+1} \) and this is the number it was our purpose to determine.

The University of California

* If we define \( b_h \) as the number of secant lines of a \( V_h^n \) of an \( S_{2k+2} \) that meet a given line of \( S_{2h+2} \), we see that \( b_0 \) is the number of lines determined by \( n \) given points in a plane.

† We may define \( j_{h-1} \) as the number of tangent lines of a \( V_h^n \) of an \( S_{2k} \) that pass through a given point of \( S_{2k} \). Then, \( j_0 \) is the class of a plane curve which is the plane projection of the curve \( C^a \) of intersection of \( V_k^n \) and an \( S_{k+2} \).

‡ B. C. Wong, On the number of stationary tangent \( S_{h-1}'s \) to a \( V_k \) in an \( S_{2h-k-1} \), this Bulletin, vol. 39, pp. 608–610.