THE DIOPHANTINE EQUATION

\[ x_1x_2a_1 + x_2x_3a_2 + \cdots + x_nx_{n+1}a_n = \delta \]

BY E. J. FINAN

The object of this paper is to prove the following theorem.

**Theorem.** If \( \delta \) is the greatest common divisor of \( a_1, a_2, \cdots, a_n \), then the Diophantine equation

\[ (1) \quad x_1x_2a_1 + x_2x_3a_2 + \cdots + x_nx_{n+1}a_n = \delta \]

always has a solution.

We shall first prove the following lemma.

**Lemma.** If the greatest common divisor of \( \alpha \) and \( \beta \) is \( \gamma \), there exist integers \( x \) and \( y \) such that \( x\alpha + y\beta = \gamma \) and such that \( x \) is prime to any previously given integer \( d \).

If \( \beta = 0 \) the lemma is evident. Hence assume \( \beta \neq 0 \). Let \( x \) and \( y \) be any pair of integers for which \( x\alpha + y\beta = \gamma \). Let \( \alpha = \alpha_1\gamma \) and \( \beta = \beta_1\gamma \). Then

\[ (2) \quad x\alpha_1 + y\beta_1 = 1. \]

This gives \( y = (1 - x\alpha_1)/\beta_1 \). Hence \( x \) must satisfy the congruence \( \alpha_1x \equiv 1 \pmod{\beta_1} \). Since \( \alpha_1 \) and \( \beta_1 \) are relative prime, all the solutions are given by \( x = x_1 + k\beta_1 \), where \( x_1 \) is a particular solution and \( k \) is an integer. Evidently all such values of \( x \) satisfy (2). Dirichlet's theorem on the infinitude of primes in an arithmetical progression assures us of the existence of a \( k \) that will satisfy the conditions of the lemma.

We shall now prove the theorem by induction. Since we may divide (1) by \( \delta \) it is quite general to assume that the \( a \)'s are relatively prime and \( \delta \) is unity.

When \( n \) is even the terms of (1) may be grouped in the following manner:

\[ (3) \quad x_2(x_1a_1 + x_2a_2) + \cdots + x_{r+1}(x_r a_r + x_{r+2}a_{r+1}) + x_{r+2}(x_{r+2}a_{r+2} + x_{r+4}a_{r+4}) + \cdots + x_n(x_{n-1}a_{n-1} + x_{n+1}a_n) = 1, \]
where \( r \) is odd. We shall now determine the \( x \)'s that will satisfy (3). Let \( \delta_i \) be the greatest common divisor of \( a_1, a_2, \ldots, a_i, \) (\( i = 1, 2, \ldots, n \)). We shall find \( x_{r+4} \) so that (a) \( x = x_{r+4}a_{r+2} + x_{r+4}a_{r+3} \) is prime to \( \delta_{r+1} \) except for factors in \( \delta_{r+3} \), and (b) \( x_{r+4} \) is prime to \( \delta_{r+3} \), if (c) \( x_{r+2} \) is prime to \( \delta_{r+1} \).

Let \( \delta_{r+3} = q_1^{a_1}q_2^{a_2} \ldots q_m^{a_m}, a_{r+3} = \delta_{r+3}f a_{r+3}', \) and \( a_{r+2} = \delta_{r+3}f a_{r+2}' \), where \( a_{r+2} \) and \( a_{r+3}' \) are relatively prime. Let \( \delta_{r+1} = \delta_{r+3}d_1, \) and \( p = p_1^{b_1}p_2^{b_2} \ldots p_k^{b_k} \) be the greatest common divisor of \( d_1 \) and \( a_{r+2}' \). Then set \( a_{r+2}' = p_1^{b_1}p_2^{b_2} \ldots p_k^{b_k}a_{r+2}', \) \( d_1 = pd_2. \) But since \( d_2 \) may contain some of the \( p \)'s, write \( d_2 = p_1^{c_1}p_2^{c_2} \ldots p_k^{c_k}d_3, \) where \( d_3 \) contains none of the \( p \)'s as a factor and hence is prime to \( a_{r+2}' \). Finally let \( d_3 = q_1^{i_1}q_2^{i_2} \ldots q_m^{i_m}d_4, \) so that \( d_4 \) is prime to \( \delta_{r+3}. \)

We assert that \( d_4 \) is the \( x_{r+4} \) we want. Evidently (b) is satisfied by the definition of \( d_4. \) We shall show that (a) is satisfied. We now have \( a_{r+2} = \delta_{r+3}d_3^{i_1}d_4, \) \( a_{r+3} = \delta_{r+3}d_3^{i_2}d_4, \) and \( a_{r+3} = \delta_{r+3}d_3^{i_3}d_4, \)

(\( s_i = e_i + \beta_i; \) \( i = 1, 2, \ldots, k \)). Consider

\[
\begin{align*}
x &= x_{r+2}d_{r+3}f \hat{p}_1^{b_1}p_2^{b_2} \ldots p_k^{b_k}a_{r+2}' + d_4 \delta_{r+3}f a_{r+3}'
\end{align*}
\]

and

\[
\begin{align*}
\delta_{r+1} &= \delta_{r+3}d_3^{i_1}d_4 \hat{p}_1^{b_1}p_2^{b_2} \ldots p_k^{b_k}d_3^{i_2}d_4 \ldots q_m^{i_m}d_4.
\end{align*}
\]

Evidently \( \delta_{r+3} \) is a common factor.

If \( p_i \neq q_j, \) (\( i = 1, 2, \ldots, k; j = 1, 2, \ldots, m \)), \( p_i \) cannot divide \( x \) because \( p_i \) is prime to \( d_4 \) and to \( f \) because \( \delta_{r+3} \) is the greatest common divisor of \( a_1, a_2, \ldots, a_{r+3}, \) and to \( a_{r+3}' \) because \( p_i \) divides \( a_{r+2}' \) while \( a_{r+2}' \) and \( a_{r+3}' \) are relatively prime. Also \( d_4 \) is prime to \( x \) because it is prime to \( x_{r+2} \) due to the fact that \( d_4 \) is contained in \( \delta_{r+1} \) and by (c) \( \delta_{r+1} \) is prime to \( x_{r+2}. \) Finally, \( d_4 \) is prime to \( f \) since \( \delta_{r+3} \) is the greatest common divisor of \( a_1, a_2, \ldots, a_{r+3}, \) and \( d_4 \) is prime to \( a_{r+2}' \) by definition of \( p. \) Hence (a) is satisfied, for the only factors common to \( \delta_{r+1} \) and \( x \) are the factors \( q_i \) which are all found in \( \delta_{r+3}. \)

Now by the lemma we can find \( x_1 \) and \( x_3 \) so that \( x_1a_1 + x_3a_2 = \delta_2 \) and \( x_3 \) is prime to \( \delta_2. \) Then by the method given above we can find \( x_5 \) so that \( x_3a_3 + x_5a_4 = \delta_2 \) except for factors in \( \delta_4. \)

We continue in this way until we have determined all the \( x \)'s in parentheses in (3). The quantities in these parentheses are relatively prime, for if \( g \) were a common factor it would be in \( \delta_2. \) Then according to the derivation of the \( x \)'s given above \( g \) can
appear in the second parenthesis as a factor only if $\delta_4$ contains $g$, and in the third parenthesis only if $\delta_6$ contains $g$, and so on. But since $\delta_n = 1$, $g$ must be a unit. If the quantities in parenthesis are relatively prime it is well known that we can find $x_2, x_4, \ldots, x_n$ so that (3) is satisfied.

If $n$ is odd the procedure is just the same with the exception that the single term in the last parenthesis is automatically prime to those in the preceding parentheses which have been determined as above, and as before we have only to determine $x_2, x_4, \ldots, x_n$.

The above theorem not only proves the existence of the $x$'s but also enables us to calculate them.

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ON CONTINUED FRACTIONS WHICH REPRESENT MEROMORPHIC FUNCTIONS*

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1. Introduction. In this paper I shall give sufficient conditions in order that a continued fraction of the form

$$\frac{b_1}{1 + \frac{b_2}{1 + \frac{b_3}{1 + \cdots}}}$$

with arbitrary real or complex coefficients not zero shall represent a meromorphic function of $z$. Van Vleck† has shown that a sufficient condition is the following:

$$\lim_{n \to \infty} b_n = 0.$$  

Stieltjes‡ proved that (2) is necessary as well as sufficient when the $b_n$ are real and positive. Van Vleck§ proved that when the $b_n$ are real and $b_{2n}b_{2n+1} > 0$, and the roots of the denominators, $D_{2n+1}$, of the $(2n+1)$th convergents of (1) have distinct limits, not zero, for $n = \infty$, then the condition (2) is necessary and sufficient in order that (1) shall represent a meromorphic function of

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* Presented to the Society, April 15, 1933.
† E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476-483.
‡ Stieltjes, Oeuvres, vol. 2, pp. 560-566.