1. Introduction. Let \( f(x) \) be defined in the closed interval \( I \). If \( f(x) \) has a continuous \( m \)th derivative, it can be expanded in a Taylor’s formula with \( m+1 \) terms plus remainder; the \( m \)th difference quotient of \( f(x) \) approaches \( \frac{d^m f(x)}{dx^m} \) uniformly. If \( f(x) \) is a polynomial of degree at most \( m-1 \), then the \( m \)th difference quotient is identically zero. The object of the present note is to prove converses of these theorems. The results hold also in an open interval, as they hold in every closed subinterval.

2. Difference Quotients. Given a function \( f(x) \) defined in \( I \), the \( p \)th difference quotient is defined by the equations
\[
\Delta^p_h f(x) = \frac{1}{h^p} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} f(x + ih)
\]
for \( p > 0 \). We say \( \Delta^p_h f(x) \to f_p(x) \) uniformly in \( I \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |\Delta^p_h f(x) - f_p(x)| < \varepsilon \) for every \( x \) in \( I \) and every \( h, |h| < \delta \).

(a) Suppose \( f_0(x), f_1(x), \ldots, f_m(x) \) are defined in \( I \), and
\[
(2) f_0(x + h) = f_0(x) + \frac{1}{1!} f_1(x) h + \cdots + \frac{1}{m!} f_m(x) h^m + R(x, h),
\]
where \( R(x, h)/h^m \to 0 \) uniformly in \( I \) as \( h \to 0 \). If we form the \( p \)th difference quotient, we find

\* Presented to the Society, June 23, 1933.
\* National Research Fellow.
\* We always suppose that the values of \( x \) under consideration (here, \( x + ih \) for \( i = 0, \ldots, p \)) lie in \( I \).
\* If we solve the linear equations (setting \( 0^o = 1 \))
\[ 0^j u_0 + 1^j u_1 + \cdots + p^j u_p = p! \delta_{jp}, \quad (j = 0, \ldots, p), \]
we find \( u_i = (-1)^{p-i} (\ast) \); hence \( \sum_{i=0}^{p} (-1)^{p-i} (\ast) i^j = 0, (j < p), \) and \( = p!, (j = p) \).
(3) \[ \Delta^p_h f_0(x) = \frac{1}{h^p} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} \left[ \sum_{j=0}^{m} \frac{f_j(x)}{j!} (ih)^j + R(x, ih) \right] \]

\[ = \sum_{j=0}^{m} \frac{f_j(x)}{j!} h^{i-p} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} i^j + \frac{1}{h^p} R_p(x, h) \]

\[ = f_p(x) + \sum_{j=p+1}^{m} \frac{f_j(x)}{j!} h^{i-p} \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} i^j \]

\[ + \frac{1}{h^p} R_p(x, h), \]

where

(4) \[ R_p(x, h) = \sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} R(x, ih). \]

From (2) we see that \( f_0(x) \) is continuous; hence \( \Delta^p_h f_0(x) \) is continuous (\( h \) fixed, \( p = 1, \ldots, m \)). Setting \( p = m \) in (3), we see that \( \Delta^m_h f_0(x) \to f_m(x) \) uniformly; hence \( f_m(x) \) is continuous. Having proved that \( \Delta^p_h f_0(x) \to f_p(x) \) uniformly and \( f_p(x) \) is continuous, \( f_p(x) \) is continuous, \( (q = m, m - 1, \ldots, p + 1) \), (3) shows that \( \Delta^q_h f_0(x) \to f_q(x) \) uniformly, and hence \( f_p(x) \) is continuous, \( (p = 1, \ldots, m) \).

(b) Suppose \( f(x) = f_0(x) \) and \( d^p f(x)/dx^p \) exists and equals the continuous function \( f_p(x) \), \( (p = 1, \ldots, m) \). Then (2) holds, by Taylor’s theorem, and hence \( \Delta^p_h f(x) \to d^p f(x)/dx^p \) uniformly, \( (p = 1, \ldots, m) \).

(c) Suppose \( f(x) = a_0 + a_1 x + \cdots + a_m x^m \) is a polynomial of degree at most \( m \). Then a Taylor’s formula (2) holds with \( f_m(x) \equiv m! a_m \) and \( R(x, h) \equiv 0 \), and using (3) with \( p = m \), we have \( \Delta^m_h f(x) \equiv m! a_m \). If \( f(x) \) is of degree at most \( m - 1 \), \( \Delta^m_h f(x) \equiv 0 \).

3. Rolle’s Theorem for Difference Quotients. We shall prove first the following lemma.

**Lemma 1.** If \( k_0 < k_1 < \cdots < k_m \) are integers, \( h > 0 \), and

\[ (-1)^{m-i} f(x + kh) \geq 0, \quad (i = 0, \ldots, m), \]

then there is an integer \( k_0^{(m)} \) \( (k_0^{(m)} < k_m) \) such that

\[ \Delta^m_h f(x + k_0^{(m)} h) \geq 0. \]

As \( f(x + k_m h) \geq 0 \) and \( f(x + k_{m-1} h) \leq 0 \), there is a \( k_{m-1}^{(1)} \),
Taylor's Formula

\( k_{m-1} \leq k_{m-1}^{(1)} < k_m \), such that \( \Delta^1_k f(x + k_{m-1}^{(1)} h) \geq 0 \). In this manner we find numbers \( k_0^{(1)}, \ldots, k_{m-1}^{(1)} \), \( k_{m-1}^{(1)} < k_i \), such that

\[ (-1)^{m-1-i} \Delta^1_k f(x + k_i^{(1)} h) \geq 0, \quad (i = 0, \ldots, m-1). \]

By the same method, using \( \Delta^1_k f(x) \) in place of \( f(x) \), we find numbers \( k_0^{(2)}, \ldots, k_{m-2}^{(2)}, (k_{m-1}^{(1)} \leq k_{m-2}^{(2)} < k_i^{(1)}) \), such that

\[ (-1)^{m-2-i} \Delta^2_k f(x + k_i^{(2)} h) \geq 0, \quad (i = 0, \ldots, m-2). \]

Continuing in this manner we find, finally, the required number \( k_0^{(m)} \).

4. Theorem 1. Let \( f(x) \) be measurable in the closed interval \( I \). A necessary and sufficient condition that \( f(x) \) be a polynomial of degree at most \( m-1 \) is that \( \Delta^m_k f(x) \to 0 \) uniformly.*

The theorem is not true without the measurability condition. For the discontinuous functions defined by G. Hamel† are seen to have the property \( \Delta^m_k f(x) = 0 \).

The necessity of the condition follows from §2 (c); we must prove the sufficiency. We shall prove first the following lemma.

Lemma 2. Let \( f(x) \) satisfy the conditions of the theorem. Let \( r_1, r_2, \ldots, r_m \) be rational numbers. Take a fixed number \( a \) and a fixed \( t \). Let \( P(x) \) be the polynomial of degree at most \( m-1 \) such that \( f(a + r_i t) = P(a + r_i t), (i = 1, \ldots, m) \). Then if \( r \) is any rational number, \( f(a + rt) = P(a + rt) \).

Suppose that the lemma is not true. Then for some rational \( r_0, f(a + r_0 t) \neq P(a + r_0 t) \). Let

\[ Q(x) = c_0 + c_1 x + \cdots + c_m x^m, \quad (c_m \neq 0), \]

be the polynomial of degree \( m \) such that \( Q(a + r_i t) = f(a + r_i t), (i = 0, \ldots, m) \). Set \( \phi(x) = f(x) - Q(x) \); then \( \phi(a + r_i t) = 0, (i = 0, \ldots, m) \), and by §2 (c),

\[ \Delta^m_h \phi(x) = \Delta^m_h f(x) - m! c_m \to - m! c_m \]

uniformly as \( h \to 0 \). Hence we can take a \( \delta > 0 \) so that \( \Delta^m_h \phi(x) < 0 \) or \( > 0 \) for all \( x \) in \( I \) and \( h < \delta \) according as \( c_m > 0 \) or \( c_m < 0 \). As

---

* The case that \( f(x) \) is continuous and \( \Delta^m_k f(x) = 0 \) has been considered by Anghelutza, Mathematica (Cluj), vol. 6 (1932), pp. 1–7.

† Mathematische Annalen, vol. 60 (1905), p. 461, equation (2).
the $r_i$ are rational, there is a number $r$ such that $rt<\delta$ and the numbers $a+krt$ ($k$ an integer) include the numbers $a+ri$. By Lemma 1, there are integers $k'$ and $k''$ such that $\Delta_{ni}\phi(a+k'rt) \geq 0$ and $\Delta_{ni}
phi(a+k''rt) \leq 0$, a contradiction, proving the lemma.

5. Proof of Theorem 1. If we define $P(x)$ as in the lemma, $a$ and $t$ being rational, then $f(x)=P(x)$ at all rational points of $I$. If $f(x)$ is continuous, it follows that $f(x)=P(x)$ in $I$. To complete the proof of the theorem, we must show that if $f(x) \neq P(x)$, then $f(x)$ is not measurable.

Suppose there is a number $a$ in $I$ such that $f(a) \neq P(a)$. If $Q(x)$ is the polynomial of degree at most $m-1$ such that $f(x)=Q(x)$ at points $a, a+r_1, \ldots, a+r_m$, ($r_1, \ldots, r_m$ rational), then $f(x)=Q(x)$ at all points $a+r$, ($r$ rational), by the lemma. Set $\sigma = |Q(a)-P(a)|$, and take $\delta>0$ so that

\[ (7) \quad |P(x)-P(a)| < \frac{\sigma}{4}, \quad |Q(x)-Q(a)| < \frac{\sigma}{4}, \quad (|x-a| \leq \delta). \]

Take $\eta>0$ so that if $R(x)$ is any polynomial of degree at most $m-1$ such that $|R(i)| \leq \eta$, ($i=1, \ldots, m$), then $|R(0)| < \sigma/4.*$ By a change of variable, we see then that for any $x, h$, and $y_0$, if $|R(x+ih)-y_0| \leq \eta$, ($i=1, \ldots, m$), then $|R(x)-y_0| < \sigma/4$.

If $f(x)$ is measurable, there is a number $y_0$ such that the set of points $E$ in $(a-\delta, a+\delta)$ (and in $I$) for which $|f(x)-y_0| \leq \eta$ is of positive measure. Let $I'$ be a closed subinterval of $(a-\delta, a+\delta)$ such that if $E_m=E \cdot I'$, then $\dagger$

\[ (8) \quad \text{meas (} E_m \text{)} > \frac{m^2}{m^2+1} \text{ meas (} I' \text{)}. \]

Either $|y_0-P(a)| \geq \sigma/2$ or $|y_0-Q(a)| \geq \sigma/2$, say the latter. Let $b$ be a number in $I'$ such that $f(b)=Q(b)$, (take $b-a$ rational); then, using (7), $|f(b)-y_0| \geq \sigma/4$.

Let $S$ be the set of numbers $s$ such that $b+ms$ is in $E_m$, and $E_i$ the set of numbers $b+is$, where $s$ is in $S$, ($i=1, \ldots, m-1$).

* That this can be done follows easily from the fact that $\Delta_{ni} R(0)=0$; see §2 (c).

Set $E_i' = E_i - E_m$, and let $S_i$ be the set of numbers $s$ in $S$ such that $b + is$ is in $E_i'$, $(i = 1, \ldots, m)$. There is no number $s_0$ in every $S_i$. For if there were, we would have simultaneously

$$(9) \quad |f(b + is_0) - y_0| \leq \eta, \quad (i = 1, \ldots, m), \quad |f(b) - y_0| > \sigma/4.$$ 

Hence by our choice of $\eta$, if $R(x)$ is the polynomial of degree at most $m-1$ such that $R(b + is_0) = f(b + is_0), (i = 1, \ldots, m)$, then $R(b) \neq f(b)$. But this contradicts Lemma 2.

Consequently, every number $s$ of $S$ is in some set $S - S_i$, and hence for some $j$, $\text{meas } (S - S_i) \geq \text{meas } (S)/m$. Therefore

$$\text{meas } (E_i - E_j') \leq \frac{\text{meas } (E_i)}{m} \geq \frac{\text{meas } (E_m)}{m^2}.$$ 

As $E_m$ and $E_i - E_j'$ have no common points, this with (8) gives

$$(10) \quad \text{meas } [E_m + (E_j - E_j')] \geq \text{meas } (E_m) \left[1 + \frac{1}{m^2}\right]$$

$$> \text{meas } (I').$$

But this contradicts the fact that $E_m + E_i$ lies in $I'$. Hence $f(x)$ is not measurable, and the theorem is proved.

**Theorem 2.** Let $f(x)$ be measurable in the closed interval $I = (\alpha, \beta)$. A necessary and sufficient condition* that $d^m f(x)/dx^m$ exist and equal $f_m(x)$ is that $\Delta^m f(x) \to f_m(x)$ uniformly in $I$.

---

* A direct proof of this theorem may be given as follows, as suggested by Birkhoff. The function $f(hk), (h = 1/p, k \text{ and } p \text{ integral})$, may be expressed in terms of $\Delta^m f(x)$ by the formula

$$f(hk) = \sum_{i=0}^{m-1} a_i (hk)^i$$

$$+ \frac{h^m}{(m-1)!} \sum_{i=0}^{k-1} (k - l - 1)(k - l - 2) \cdots (k - l - m + 1) \Delta^l f(lh),$$

where the $a_i$ are determined so that $f(sph) = f(s), (s = 0, \ldots, m - 1)$. We solve a set of linear equations with a Vandermonde determinant, and find

$$a_i = \sum_{s=0}^{m-1} \eta_s \left[ f(s) - \frac{h^m}{(m-1)!} \sum_{l=0}^{s-1} (sp - l - 1) \cdots (sp - l - m + 1) \Delta^l f(lh) \right].$$

---
The necessity of the condition was proved in §2 (b). To prove the sufficiency, we show first that $f_m(x)$ is continuous. Take $\delta$ so small that $|\Delta^m f(x) - f_m(x)| < \epsilon/2$ for $|h| < \delta$. Take any two points $x_1$ and $x_2$ of $I$ such that $x_2 - x_1 = mh$, $|h| < \delta$. Then $|\Delta^m f(x_1) - f_m(x_1)| < \epsilon/2$, $|\Delta^m f(x_2) - f_m(x_2)| < \epsilon/2$. But $\Delta^m f(x_2) = \Delta^m f(x_1)$; hence $|f_m(x_2) - f_m(x_1)| < \epsilon$.

Define the $m$-fold integral

$$g(x) = \int_a^x \cdots \int_a^{x_2} \int_a^{x_3} \cdots \int_a^{x_m} f_m(x_1)dx_1dx_2 \cdots dx_m;$$

then $d^mg(x)/dx^m = f_m(x)$ in $I$, and by §2 (b), $\Delta^m g(x) \rightarrow f_m(x)$ uniformly in $I$. Set $\phi(x) = f(x) - g(x)$. Then $\Delta^m \phi(x) \rightarrow 0$ uniformly; hence $\phi(x)$ is a polynomial of degree at most $m - 1$ (Theorem 1), and therefore $f(x)$ has a continuous $m$th derivative in $I$, and

$$d^m f(x)/dx^m = d^m \phi(x) + d^m g(x) = f_m(x).$$

**Theorem 3.** A necessary and sufficient condition that the function $f(x) = f_0(x)$ defined in $I$ have a continuous $m$th derivative is that there exist functions $f_1(x), \ldots, f_m(x)$ such that (2) holds. In this case, $d^pf(x)/dx^p = f_p(x)$, $(p = 1, \ldots, m)$. The necessity of the condition is a consequence of Taylor's formula. To prove the sufficiency, §2 (a) shows that $\Delta^m f(x) \rightarrow f_p(x)$ uniformly as $h \rightarrow 0$; consequently†, by Theorem 2, $d^pf(x)/dx^p$ exists and equals $f_p(x)$, $(p = 1, \ldots, m)$.

Harvard University

where the $\eta_{i}$ are numerical coefficients. If in the above equations we replace $p$ by $p_i = 2^i p$, $h$ by $h_i = h/2^i$, and $k$ by $k_i = 2^i k$, and set $r = k_j h_j = k h$, then as $\Delta^m f(x) \rightarrow f_m(x)$ uniformly, passing to the limit $j = \infty$ gives

$$r = \sum_{i=0}^{n-1} \sum_{s=0}^{n-1} \eta_{is} \left[ f(s) - \frac{1}{(m - 1)!} \int_0^s (s - t)^{m-1} f_m(t)dt \right]$$

$$+ \frac{1}{(m - 1)!} \int_0^r (r - t)^{m-1} f_m(t)dt,$$

for any rational $r > 0$. If $f(x)$ is continuous, this is true for all $r > 0$ and, the theorem follows immediately on differentiating. (The restriction $r > 0$ may of course be replaced by the restriction $r > -c$ for any $c$.)

† A direct proof of this theorem is given by the author in a paper entitled *Differentiable functions defined in closed sets I*, Theorem IV.