ENUMERATIVE PROPERTIES OF CURVES

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In the determination of enumerative properties of algebraic curves it is often convenient to decompose a given curve $C^n$ of order $n$ and genus $p$ to be studied into a number of component curves the sum of whose orders is equal to $n$. We may decompose $C^n$ in various ways but we find it most convenient to decompose it completely into $n$ lines with $n-1+p$ incidences. We call the system formed by these $n$ lines an $n$-line or a skew $n$-sided polygon $\Gamma$ with $n-1+p$ vertices. To determine the enumerative properties of the given curve $C^n$, we, in this paper, determine certain enumerative properties of $\Gamma$ and then interpret the results for $C^n$. We shall obtain in this manner a number of results for $C^n$ some of which are already well known and the others are less well known or are new.

Let the symbol $\{n\}_{x_1}^{(s)} x_2 \ldots x_q$ denote the number of groups each consisting of $x_1+x_2+\ldots+x_q$ sides which are arranged in $q$ sets such that each set contains $x_i$ consecutive sides and that any two sets are separated by at least $s$ consecutive sides not contained in them. Thus, $\{n\}_{11}^{(1)}$ means the number of pairs of non-consecutive sides of $\Gamma$. If $q = 1$, we have $\{n\}_{x_1}^{(s)}$ or just $\{n\}_{x_1}$, which is the number of groups each of $x_1$ consecutive sides. The symbol $\{n\}_1^{(s)}$ or $\{n\}$ means the number of groups each containing no members and is therefore equal to unity. Hence,

$$\{n\}_1^{(s)} = \{n\}_1 = 1. \tag{1}$$

The following formula can be easily verified or can be proved by the method used below:

$$\{n\}_s^{(s)} = \{n\}_{x_1} = n - (x_1 - 1) + (x_1 - 1)p. \tag{2}$$

The number of groups each consisting of $q$ pairs of intersecting sides (or the number of groups of $q$ non-consecutive vertices) of $\Gamma$ is known† and is given by

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* Presented to the Society, November 28, 1931.
† This result is given without proof by B. C. Wong, *On loci of $(r-2)$-spaces incident with curves in $r$-space*, this Bulletin, vol. 36 (1930), pp. 755–761.
The formula for the number of groups of $q$ non-consecutive sides of $T$ is, as we shall see,

$$\{n\}^{(0)}_{x_1,x_2,\ldots, x_q} = \sum_{i=0}^{q} \binom{n - q - i}{q - i} \binom{\nu}{i}.$$  

(3)

where $\nu = q/2$ for $q$ even and $\nu = (q - 1)/2$ for $q$ odd.

Now we proceed to the determination of $\{n\}^{(1)}_{x_1,x_2,\ldots, x_q}$. We may assume

$$\{n\}^{(1)}_{x_1,x_2,\ldots, x_q} = M + N - N',$$

where $M$ is the required number when $\nu = 0$ and $N$ and $N'$ are functions of $n$ and $\nu$, each containing $\nu$ as a factor.

To find the value of $M$, let $\Gamma$ be an open polygon, that is, a system of $n$ lines $l_1, l_2, \ldots, l_n$ such that $l_1$ meets $l_2$, $l_2$ meets $l_3$, $\ldots$, $l_{n-1}$ meets $l_n$ but $l_n$ is skew to $l_1$. This polygon has $n - 1$ vertices, the minimum number it can have without becoming composite. By the simple process of counting, we find

$$M = q! \left(\frac{n - \Sigma x_i - q s + q + s}{q}\right).$$

This is verified for $n = \Sigma x_i + q s - s$, which gives $M = q!$.

To determine $N$, we notice that $\nu$ is the number of additional vertices $\Gamma$ may possess over and above the minimum, $n - 1$. Let a general $\Gamma$ with $n - 1 + \nu$ vertices be given. Consider two series of consecutive sides $a_1, a_2, \ldots, b_1, b_2, \ldots$, respectively. If all the $a$'s are skew to all the $b$'s, then $a_1, a_2, \ldots, a_{x_i}$ and $b_1, b_2, \ldots, b_{x_j}$ form two distinct sets and so do $b_1, b_2, \ldots, b_{x_i}$ and $a_1, a_2, \ldots, a_{x_j}$. Now let $a_1$ and $b_1$ meet. The point of meeting is then one of the $\nu$ additional vertices of $\Gamma$ over and above the minimum. For each additional vertex we have $x_i - 1$ additional sets given by $a_{\lambda}, a_{\lambda-1}, \ldots, a_1, b_1, b_2, \ldots, b_{x_i-\lambda}$, ($\lambda = 1, 2, \ldots, x_i - 1$). Since each of these sets is to be combined with $q - 1$ other sets chosen from the $(n - x_i - 2s)$-sided polygon.
Γ' obtained from Γ with the \( x_i + 2s \) consecutive sides \( a_\lambda + s, a_\lambda + s - 1, \ldots, a_1, b_1, b_2, \ldots, b_{q - \lambda - s} \) removed, we have

\[
N = \binom{p}{1} \sum_{i=0}^{q} (x_i - 1) \left\{ n - x_i - 2s \right\}_{x_1 x_2 \ldots x_q / x_i}^{(e)}.
\]

Now it is necessary to deduct a certain number \( N' \) of groups arising from the supposition that \( a_1 \) and \( b_1 \) are incident. Every group containing a pair of sets \( a_{\alpha + 1}, a_{\alpha + 2}, \ldots, a_{\alpha + x_1}; b_{\beta + 1}, b_{\beta + 2}, \ldots, b_{\beta + x_1} \), where \( 0 \leq \alpha + \beta \leq s - 1 \), is to be deducted, as these two sets are separated by only \( \alpha + \beta \) sides, namely, \( a_\alpha, a_{\alpha - 1}, \ldots, a_1, b_1, b_2, \ldots, b_\beta \). These two sets are to be combined with \( q - 2 \) other sets chosen from the \( (n - x_i - x_j - 2s - \alpha - \beta) \)-sided polygon obtained from Γ with the \( \alpha + \beta + x_i + x_j + 2s \) sides \( a_1, a_2, \ldots, a_\alpha, \ldots, a_{\alpha + x_1}, a_{\alpha + x_1 + 1}, \ldots, a_{\alpha + x_1 + s}, b_1, b_2, \ldots, b_\beta, \ldots, b_{\beta + x_1}, \ldots, b_{\beta + x_1 + s} \) removed. Hence, the number of groups containing these two particular sets is, for all different values of \( i \) and \( j \) from 1 to \( q \),

\[
\sum_{i \neq j} (n - x_i - x_j - 2s - \alpha - \beta)_{x_1 x_2 \ldots x_q / x_i x_j}^{(e)}.
\]

By interchanging the \( a \)'s and the \( b \)'s we have twice this number; and by allowing \( \alpha \) and \( \beta \) to take on all the permissible values, we have, remembering the factor \( p \),

\[
N' = 2 \binom{p}{1} \sum_{k=1}^{s} k \sum_{i \neq j} (n - x_i - x_j - 2s + 1 - k)_{x_1 x_2 \ldots x_q / x_i x_j}^{(e)}.
\]

Then, the result for Γ which we started out to derive is the recursion formula

\[
(A) \left\{ n \right\}_{x_1 x_2 \ldots x_q}^{(e)} = q! \left( n - \Sigma x_i - qs + q + s \right)_{q}
\]

\[
+ \binom{p}{1} \sum_{i=1}^{q} (x_i - 1) (n - x_i - 2s)_{x_1 x_2 \ldots x_q / x_i}^{(e)}
\]

\[
- 2 \binom{p}{1} \sum_{k=1}^{s} k \sum_{i \neq j} (n - x_i - x_j)_{x_1 x_2 \ldots x_q / x_i x_j}^{(e)}
\]

\[
- 2s + 1 - k)_{x_1 x_2 \ldots x_q / x_i x_j}^{(e)}.
\]
In the derivation of this formula we have tacitly assumed that all the $x$'s are different. If $t$ of the $x$'s are alike, the right-hand member is to be divided by $t!$. The following result, obtained in somewhat the same manner as above, is more general:

\[
(B) \left \{ \begin{array}{l}
\{ n \}^{(s)}_{x_1^s x_2^s \ldots x_p^s} \\
\end{array} \right.
\]

\[
= \frac{(t_1 + t_2 + \cdots + t_p)!}{t_1 t_2 \cdots t_p!} \left( n - \sum_i t_i x_i - s \sum_i t_i + \sum_i t_i + s \right) \\
+ \left( \begin{array}{c}
p \\
1 \end{array} \right) \sum_{i=1}^{p} (x_i - 1) \left\{ n - x_i - 2s \right\}^{(s)}_{x_1^s x_2^s \ldots x_p^s} \\
- \left( \begin{array}{c}
p \\
1 \end{array} \right) \sum_{k=1}^{s} k \sum_{i=1}^{p} \left\{ n - 2x_i - 2s + k - 1 \right\}^{(s)}_{x_1^s x_2^s \ldots x_p^s} \\
- 2 \left( \begin{array}{c}
p \\
1 \end{array} \right) \sum_{k=1}^{s} k \sum_{i \neq j} \left\{ n - x_i - x_j - 2s \right\}^{(s)}_{x_1^s x_2^s \ldots x_p^s} \\
+ 1 - k \right\}^{(s)}_{x_1^s x_2^s \ldots x_p^s}.
\]

This gives the number of all the groups each consisting of $x_1^s + x_2^s + \cdots + x_p^s$ sides arranged in $t_1 + t_2 + \cdots + t_p$ sets such that $t_1$ of these sets contain each $x_1$ consecutive sides, $t_2$ contain each $x_2$ consecutive sides, etc., and such that all the sets are separated from one another by at least $s$ sides.

In the expansion of these formulas, replace $(x)^{(s)}$ wherever it appears by $(x+\mu)^{(s)}$. The necessity of this replacement is due to the nature of the combinatorial ideas involved or can be easily shown by calculation of known cases.

Putting $s = 0$, $\nu = 1$, $x_1 = 2$ in (B) or putting $s = 0$, $x_1 = x_2 = \cdots = x_q = 2$ in (A) and dividing by $q!$, we have formula (3). For $s = 1$, $x_1 = x_2 = \cdots = x_q = 1$, (A) gives $q!$ times (4), and for $s = 1$, $\nu = 1$, $x_1 = 1$, (B) gives exactly the result (4).

Now we are going to derive a few results for curves. Consider a general curve $C^n$ in $S_3$. Since two skew lines have one apparent intersection or apparent double point, the number of apparent double points of $C^n$ is equal to the number of pairs of non-intersecting sides of $\Gamma$ and is therefore given by

\[
\left\{ n \right\}^{(1)}_{11} = \binom{n-1}{2} - p.
\]
The plane of a conic taken twice may be regarded as its developable. Hence, the order of the developable surface of $C^n$ is twice the number of pairs of consecutive sides of $\Gamma$ and is, therefore,

$$2\{n\}_2 = 2(n - 1 + p).$$

Similarly, the developable $V_m$ of $C^n$ in $S_r$ is of order

$$m\{n\}_m = m[n - (m - 1) + (m - 1)p].$$

Also, the number of hyperosculating hyperplanes of $C^n$ passing through a given point of $S_r$ is

$$r\{n\}_r = r[n - (r - 1) + (r - 1)p],$$

and the number of hyperstationary tangent $S_{r-1}$'s is

$$(r + 1)\{n\}_{r+1} = (r + 1)(n - r + rp).$$

The number of quadrisecant lines of a $C^n$ in $S_3$ is obtained from the following consideration. Any four general skew lines have two transversals. If they intersect in pairs or if two of them intersect only, there is only one transversal. Hence, the required number is

$$2\{n\}_4^{(1)} + \{n\}_4^{(1)} + \{n\}_2^{(1)}.$$

It is not difficult to see that the same number is also given by

$$2\binom{n}{4} - \binom{n - 2}{2} \{n\}_2 + \{n\}_2^{(1)}.$$

Either case yields the same required expression

$$\frac{1}{12} (n - 2)(n - 3)^2(n - 4) - \frac{1}{2} (n - 3)(n - 4)p + \frac{1}{2} p(p - 1).$$

In the same manner, the order of the surface of trisecant lines is found to be

$$2\{n\}_3^{(1)} + \{n\}_3^{(1)} = \frac{1}{3} (n - 1)(n - 2)(n - 3) - (n - 2)p.$$
planes of $C^n$ in $S_3$ is 8 times the number of planes each containing three non-consecutive vertices of $\Gamma$ and is therefore

$$8\{n\}_{222}^{(0)} = 8\left[\binom{n-3}{3} + \binom{n-4}{2}\binom{p}{1} + \binom{n-5}{1}\binom{p}{2} + \binom{p}{3}\right].$$

Similarly, the number of hyperplanes tangent to a $C^n$ in $S_r$ at $r$ distinct points is given by

$$2^r\{n\}_{2r}^{(0)} = 2^r \sum_{i=0}^{r} \binom{n-r-i}{r-i}\binom{p}{i}.$$

As we have already given examples sufficiently numerous to illustrate the method used, we shall record just one or two more results without explanation, the symbols themselves being indicative of the meaning.

The number of $(r+1)$-secant $(r-1)$-spaces of a $C^n$ in a $2r$-space $S_{2r}$ is

$$\{n\}_{r+1}^{(1)} = \sum_{i=0}^{t} (-1)^i \binom{n-r-2i}{r+1-2i}\binom{p}{i},$$

where $t=(r+1)/2$ if $r$ is odd and $t=r/2$ if $r$ is even.

As another result we have the order of the $V_2$ formed by the quinti-secant planes of a $C^n$ in $S_4$ given by

$$5\binom{n}{5} - 2\binom{n-2}{3}\{n\}_{2}^{(0)} + \binom{n-4}{1}\{n\}_{22}^{(0)}$$

or

$$5\{n\}_{11111}^{(1)} + 3\{n\}_{21111}^{(1)} + 2\{n\}_{2211}^{(1)} + \{n\}_{23}^{(1)} + \{n\}_{311}^{(1)}.$$  

The calculation of either yields

$$(n-4)\binom{n-2}{4} - 2\binom{n-3}{3}p + (n-4)\binom{p}{2}.$$