AN INVOLUTORIAL LINE TRANSFORMATION*

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1. Introduction. Consider a non-singular quadric $H$, a plane $\pi$ not tangent to $H$, and a point $O$ on $H$ but not on $\pi$. In the plane $\pi$ take a Cremona involutorial transformation $I_n$ of order $n$ with fundamental points in general position (not necessarily on the curve of intersection of $\pi$ and $H$). Project $H$ from $O$ upon $\pi$ by the projection $P$. The point transformation $P I_n P^{-1}$ is involutorial and leaves $H$ invariant as a whole. A point $A$ on $H \sim (P) B$ on $\pi; \parallel B \sim (I_n) B'; B' \sim (P^{-1}) A'$ on $H$. Now an arbitrary line $t$, with Plücker coordinates $y_i$, $(i = 1, \cdots, 6)$, meets $H$ in two points $A_1, A_2$ which $\sim (P I_n P^{-1}) A'_1, A'_2$. The line $A'_1 A'_2 \equiv t'$ shall be called the conjugate of $t$ by the line transformation $T$, and we write $t \sim (T)t'$. Since the point transformation $P I_n P^{-1}$ is involutorial, so will the line transformation $T$ be involutorial.

2. Order of the Transformation $T$. The coordinates of the points $A_1, A_2$ in which $t$ meets $H$ are quadratic functions of $y_i$; the coordinates of $B_1, B_2$ are linear in the coordinates of $A_1, A_2$ and hence are also quadratic functions of $y_i$; the coordinates of $B'_1, B'_2$ are functions of degree $n$ in the coordinates of $B_1, B_2$ and are therefore functions of degree $2n$ in $y_i$; finally $A'_1, A'_2$ have coordinates of degree $2n$ in $y_i$. The Plücker coordinates of a line are quadratic functions of the coordinates of two points which determine the line, and hence the Plücker coordinates $x_i$ of $t'$ are of degree $4n$ in $y_i$. Thus $T$ is of order $4n$.

3. The Singular Lines of $T$. Denote by $O_1, O_2$ the points where the generators $g_1, g_2$ of $H$ through $O$ meet $\pi$. The points $O_1, O_2 \sim (I_n) O'_1, O'_2 \sim (P^{-1}) Q_1, Q_2$. The line $t \equiv Q_1 Q_2 \sim (T)$ the entire plane field of lines $(g_1 g_2)$, since $O_1, O_2 \sim (P^{-1}) g_1, g_2$.

Any line $t$ tangent to $H$ meets $H$ in two points coincident at $A$. The point $A \sim (P I_n P^{-1}) A'$, and hence $t \sim (T)$ the pencil of tangents to $H$ at $A'$.

Since $O \sim (P)$ the whole line $O_1 O_2 \sim (I_n)$ a curve $\rho$ of order

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† The symbol $\sim (P)$ means "corresponds in the transformation $P$ to."
n \sim (P^{-1}) \), a curve of order \(2n\) with an \(n\)-fold point at \(O\), any line through \(O\) meeting \(H\) again at \(A \sim (T)\) a cone of order \(2n\) with vertex \(A'\) and an \(n\)-fold generator \(A'O\). However, when \(t\) is tangent to \(H\) at \(O\) so that both points of intersection with \(H\) coincide there, then \(t \sim (T)\) a congruence of lines, bisecants of the curve of order \(2n\) into which \(\rho\) is projected by \(P^{-1}\). The order of the congruence is the number of bisecants through an arbitrary point of space, and hence the number of apparent double points of the curve. Since \(\rho\) is rational and since also its projection on \(H\) by \(P^{-1}\) is rational, we have, from an arbitrary point of space,

\[
\frac{(2n-1)(2n-2)}{2} - \frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{2} = n(n-1)
\]

apparent double points, and hence the conjugate congruence is of order \(n(n-1)\). The class is the number of bisecants lying in an arbitrary plane, which is \(n(2n-1)\).

Denote the regulus to which \(g_1\) belongs by \(k_1\) and that to which \(g_2\) belongs by \(k_2\). A line \(t\) belonging to \(k_1 \sim (P)\) a line through \(O_2\) which line \(\sim (I_n)\) a curve of order \(n \sim (P^{-1})\) a curve of order \(2n\) on \(H\). Again we find that \(t \sim (T)\) a congruence of order \(n(n-1)\) and class \(n(2n-1)\). So also for any line of the regulus \(k_2\).

The line \(t=g_1 \sim (P)O_1 \sim (I_n)O'_1 \sim (P^{-1})Q_1\), and hence \(t \sim (T)\) the pencil of tangents to \(H\) at \(Q_1\) and likewise \(t=g_2(T)\) the pencil of tangents to \(H\) at \(Q_2\).

4. **The Invariant Lines of \(T\).** Let the curve of invariant points of \(I_n\) be \(\Delta_m\) of order \(m\) and genus \(p\). Then \(\Delta_m \sim (P^{-1}) \delta_{2m}\) of order \(2m\) and also of genus \(p\). Any bisecant of \(\delta_{2m}\) is invariant under \(T\), and hence the invariant lines form a congruence of order \(m(m-1)-p\) and of class \(m(2m-1)-p\). If \(I_n\) has \(q\) isolated invariant points \(R_1, R_2, \ldots, R_q\), they \(\sim (P^{-1})\) \(q\) points \(S_1, S_2, \ldots, S_q\) on \(H\), and hence there are \(qC_2=g(q-1)/2\) additional invariant lines of \(T\).

5. **Special Cases of \(T\) when \(n=1\).** Choose \(I\) as the harmonic homology with center \(R\) and axis \(\Delta\). By taking \(R\) and \(\Delta\) in general position in \(\pi\), we produce the desired results by replacing \(n\) by the number one in the foregoing paragraphs. It is only when we choose \(R\) and \(\Delta\) in special positions with regard to \(O_1\), \(O_2\) that the results must be altered.
Let $\Delta$ be the line $O_1O_2$. The order of $T$ is 4. Since each point of $\Delta$ is invariant under $I$, $O_1$, $O_2 \sim (I)$ $O_1$, $O_2 \sim (P^{-1})$ $g_1$, $g_2$. Hence every line of the plane field $(g_1g_2) \sim (T)$ the whole plane field $(g_1g_2)$.

Any line $t$ through $O$, meeting $H$ at a second point $A \sim (T)$ the two pencils $A'g_1$, $A'g_2$. A line $t$ tangent to $H$ at $O \sim (T)$ the plane field of lines $(g_1g_2)$.

A line $t$ of the regulus $k_1 \sim (P)$ a line $m$ in $\pi$ through $O_2 \sim (I)$ another line $m'$ through $O_2 \sim (P^{-1})$ another generator $m_1$ belonging to $k_1$, and thus $t \sim (T)$ the plane field $(m_1g_2)$. Likewise a line $t$ belonging to the regulus $k_2 \sim (T)$ an entire plane field of lines.

The entire plane field $(g_1g_2)$ and the bundle $(O)$ are invariant as well as singular under $T$.

Now choose $R$ at $O_1$ and $\Delta$ in general position in $\pi$. Each line through $R$ in $\pi$ is invariant as a whole under $I$, and in particular

$$O_1O_2 \sim (I)O_1O_2; \quad O_1 \sim (I)O_1; \quad O_2 \sim (I)B'_2$$

on $O_1O_2$. Any line $t$ lying in the plane $g_1g_2$ meets $g_1$, $g_2$ in points $A_1$, $A_2$ which points $\sim (P)O_1$, $O_2 \sim (I)O_1$, $B'_2 \sim (P^{-1})g_1$, $O$; hence $t \sim (T)g_1$. Since $T$ is involutorial, $g_1 \sim (T)$ the plane field $(g_1g_2) \cdot t \equiv g_2 \sim (T)$ the pencil of tangents to $H$ at $O$.

Any line $t$ belonging to the regulus $k_2 \sim (T)$ the whole plane field $(tg_1)$. Thus the regulus $k_2$ is invariant as well as singular under $T$. Any line $t$ belonging to the regulus $k_1 \sim (P)$ a line $m$ through $O_2 \sim (I)$ a line $m'$ through $B'_2 \sim (P^{-1})$ the conic $H$, $Om'$. Thus $t \sim (T)$ the plane field $(Om')$.

The invariant lines of $T$ consist of the plane field $(O\Delta)$, the pencil of tangents to $H$ at $O$, the generator $g_1$ and the regulus $k_2$. A like special case arises when we take $R$ at $O_2$ and $\Delta$ in general position in $\pi$. The results are readily obtained by interchanging the subscripts 1 and 2 in the discussions in the foregoing paragraphs.

By taking $R$ in general position and $\Delta$ through $O_1$ but not through $O_3$, we have a third special case of $T$ when $n = 1$. Now, the point $O_1$ is invariant under $I$ but $O_2 \sim (I)B'_2$, and $O_1O_2 \sim (I)O_1B'_2 \sim (P^{-1})$ a generator $b_2$ of the regulus $k_2$. Thus any line $t$ passing through $O$ and meeting $H$ at $A \sim (T)$ the pencils $A'b_2$, $A'g_1$. Any line $t$ tangent to $H$ at $O \sim (T)$ the plane field $(b_2g_1)$.
Any line $t$ belonging to the regulus $k_2 \sim (P)$ a line $m$ through $O_1 \sim (I)$ another line $m'$ through $O_1 \sim (P^{-1})$ another generator $m_2$ belonging to the regulus $k_2$. Thus $t \sim (T)$ the plane field $(m_2g_2)$. Any line $t$ belonging to the regulus $k_1 \sim (P)$ a line $q$ through $O_2 \sim (I)$ a line $q'$ through $B_2 \sim (P^{-1})$ the conic $H, Oq'$. Thus $t \sim (T)$ the plane field $(Oq')$.

The invariant lines of $T$ are the plane field $(O\Delta)$ and the line $OR$. Similarly we have a special case when $\Delta$ passes through $O_2$ and $R$ is in general position.

A fourth special case of $T$ when $n = 1$ is found by taking $R$ at $O_1$ and $\Delta$ through $O_2$. Both $O_1$ and $O_2$ are invariant under $I$ but the other points of $O_1O_2$ are not invariant. A line $t$ through $O$ and meeting $H$ again at $A \sim (T)$ the two pencils $A'g_1, A'g_2$. Any line $t$ tangent to $H$ at $O \sim (T)$ the plane field $(g_1g_2)$.

A line $t$ belonging to $k_2 \sim (T)$ the plane field $(g_1)$, and a line $t$ belonging to $k_1 \sim (P)$ a line $m$ through $O_2 \sim (I)$ another line $m'$ through $O_2 \sim (P^{-1})$ another generator $m_1$ of $k_1$. Thus $t \sim (T)$ the plane field $(m_1g_2)$.

The invariant lines of $T$ consist of the pencil of tangents to $H$ at $O$, the plane field $(O\Delta)$, the generator $g_1$ and the regulus $k_2$.

By choosing $n > 1$ and taking the $F$-points, the curve $\Delta$, and the $P$-curves of $I_n$ in special relation to $O_1, O_2$, we can set up a limitless number of specializations of this transformation.