ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

254. Professor A. A. Albert: *On cyclic fields.*

The author has recently given a construction of all cyclic fields of degree p^e over F of characteristic p. The theory of the structure of cyclic fields will now be made complete by his new results on the case where F has characteristic not p. The author proves that if F contains a primitive pth root of unity ζ, then a necessary and sufficient condition that Y of degree $p^{e-1}>1$ shall have cyclic over-fields Z of degree p^e is that $N_{Y|F}(\beta)=\zeta$ for some β in Y, and gives a construction of all such fields Z. The case where ζ is not F is then treated. (Received July 19, 1934.)

255. Professor A. A. Albert: *A greatest common divisor process for integers of a quadratic field.*

The author considers quadratic fields $F=R(m^{1/2})$ with m an integer having no square factors. If $m\equiv 2$, 3 (mod 4), then any two integers of F have the form $A=\pi(a+bm^{1/2})$, $B=\sigma(c+dm^{1/2})$ where we may take $(\pi,\sigma)=(a,b)=(c,d)=1$ without loss of generality. Define μ to be the g.c.d. of $\pi(a^2-b^2m)$, $\sigma(c^2-d^2m)$, $\pi\sigma(ad-bc)$, and write $\mu=PQ$ where P is the product of all the prime factors of μ dividing π. Then there exists an integer ρ such that $b\rho\equiv a$ (mod Q), $d\rho\equiv c$ (mod P), and $\rho^2=\lambda\mu+m$. It is then proved that A and B have a g.c.d. in F if and only if the quadratic form $\lambda x^2+2\rho xy+\mu y^2$ represents 1 or -1. A similar result is obtained for the case where $m\equiv 1$ (mod 4). (Received July 19, 1934.)

256. Professor A. A. Albert: *The principal matrices of a Riemann matrix.*

A Riemann matrix ω is a p by $2p$ complex matrix with an attached rational skew symmetric matrix C such that $\omega C\omega'=0$, $i\omega C\bar{\omega}'$ is positive definite. The matrix C is called a principal matrix of ω and it is important to know what are all principal matrices of ω in terms of a given C. In the present note I prove that the set of all principal matrices of ω is the set of all matrices AC where A ranges over all symmetric projectivities of ω with positive characteristic roots. More explicit formulas for A are given for the case where ω is a pure Riemann matrix of the first kind. (Received July 19, 1934.)

257. Dr. R. E. Basye: *Multiply connected sets.*

In a previous abstract (see this Bulletin, vol. 39, p. 862) the writer defined "simply connected set." In the present paper the natural extension of this definition to multiply connected sets of finite order is made. A necessary and sufficient condition that a connected subdomain of the plane be multiply connected of order n is that its complement have exactly $n-1$ bounded components. Among the applications is the following: In a plane S let N be a nonvacuous closed set, G any collection of closed sets, and H a countable subcollection (possibly vacuous) of G such that (1) the common part of each pair of elements of G is N, (2) every component of every element of $G-H$ intersects N, and (3) G^* is closed. If G^* separates a point A from a point B in S then G contains a countable subcollection K such that K^* intersects every compact continuum which contains $A+B$. (Received July 23, 1934.)

258. Dr. L. M. Blumenthal (National Research Fellow): *An application of a theorem of K. Gödel.*

Gödel has shown (Ergebnisse eines mathematischen Kolloquiums, Vienna, vol. 4 (1933), p. 16) that any quadruple of points in the euclidean space R_3 may be isometrically imbedded in the surface of a sphere. This theorem is applied to show that every quadruple of points of the three-dimensional spherical surface $S_{3,r}$ of radius r has this property also; that is, any four points of the $S_{3,r}$ may be imbedded isometrically in the surface $S_{2,r'}$ of some sphere of radius r'. (Received July 24, 1934.)

259. Dr. L. M. Blumenthal (National Research Fellow): *A decomposition of the spherical surface $S_{2,r}$ by means of tetrahedral quadruples.*

The points of the surface of a three-dimensional sphere of radius r, with the distance between two points defined as the geodesic (great circle) distance, form the spherical surface $S_{2,r}$. Four points of a semi-metric space form a tetrahedral quadruple provided the points are isometric with the vertices of a tetrahedron in the euclidean space R_3. The following theorem is proved: Corresponding to each triple of points p, q, r of $S_{2,r}$ there is a decomposition of $S_{2,r}$ into three sets of points: the set (x) for which p, q, r, x form a non-degenerate tetrahedral set; the set; (y) for which p, q, r, y form a degenerate tetrahedral quadruple; and the set (z) for which the points p, q, r, z are not congruent to four points of any euclidean space. If the points p, q, r lie on a great semi-circle, the set (y) is, of course, vacuous, while the set (x) consists merely of the points of the great circle containing p, q, r minus the arc $p'r'$, where p' and r' are the points diametral to p and r, respectively. (We suppose that q lies between p and r.) If p, q, r do not lie on a great semi-circle, the set (x) consists of a curve with two branches, bounding the sets (y) and (z). The nature of this curve is studied. (Received July 13, 1934.)

260. Professor H. R. Brahana: *Concerning the order of a linear transformation with coefficients in a $GF(p)$.*

Let the transformation be defined by the m-rowed square matrix X whose mth invariant factor is I_m. Then the period of the transformation is the same as the exponent to which I_m belongs, mod p. There is also obtained an expression for the order in terms of the zeros of I_m. (Received July 30, 1934.)

261. Professor H. R. Brahana: *Metabelian groups and pencils of bilinear forms.*

We consider the metabelian groups $G=\{H, U\}$ of order p^{n+m} with centrals of order p^{n-k} in which H is a maximal invariant abelian subgroup of order p^n and type $1, 1, \cdots$ and U is abelian and of type $1, 1, \cdots$. If the order of the commutator subgroup of G is p^l the problem of classification of the groups G is equivalent to the problem of classification of k-parameter (homogeneous) families of bilinear forms in l variables y and m variables x under projective transformations on the x's, the y's, and the k parameters, the coefficients of the forms and of the transformations being residues mod p. The classification is carried out in detail for $k=2$, $m=4$, in which case $l=2, 3, \cdots, 8$. (Received July 12, 1934.)

262. Professor H. B. Curry: *Foundations of the theory of abstract sets from the standpoint of combinatory logic.*

This is a formulation of Zermelo's theory of abstract sets (Mathematische Annalen, 1908) based on the author's *Functionality in combinatory logic* (see abstract 36-9-324). The object is to remove the vagueness inherent in Zermelo's notion of "definiteness." Fraenkel has done this by introducing an ad hoc definition of function; the author's general notion of function will accomplish the same end. The more formal axioms—Paarung, Vereinigung, Potenzmenge, Aussonderung, and Ersetzung (as added by Fraenkel)—here take forms of which the following are typical: Axiom der Vereinigung: $\vdash FMMS$; Axiom der Aussonderung: $\vdash F_2(FM\overline{Pr})MMT$, where M stands for "Menge," $\overline{Pr}$ for "proposition," and S and T for operators to be defined. For primitive ideas we may take M and ϵ; but the development is simpler if we take M and an abstraction operator $A\phi$, such that $A\phi$ represents "the x's such that ϕx holds." Then ϵ, and the above S, T, etc., may be defined; the definition of T is $T\equiv[\phi, x]\cdot A([y](\phi y \;\&\; y\epsilon x))$ (that is, T is the operator such that, for each ϕ and x, $T\phi x$ is the set of y's for which $y\epsilon x$ holds and ϕy is true). Four additional axioms are necessary to express fundamentals. The Axiom der Auswahl and Axiom des Unendlichen are not affected. The resulting theory is more inclusive than Fraenkel's, less so than Zermelo's. (Received August 1, 1934.)

263. Mr. John Curtiss: *Interpolation in regularly distributed points.*

Walsh has proved the following theorem (this Bulletin, vol. 38 (1932), p. 290): Let $f(z)$ be an arbitrary function integrable (R) for $|z|=1$, and let polynomials $L_n(z)$ of respective degrees $n-1$, $(n=2, 3, \cdots)$, be defined by the requirement of coinciding with $f(z)$ in the nth roots of unity; then the sequence $L_n(z)$ approaches the limit $[\int_{|z|=1}(f(t)/(t-z))dt]/(2\pi i)$ uniformly for $|z| \leqq r<1$. The present paper extends this theorem to the study of interpolation to an arbitrary function defined and integrable (R) on a Jordan curve which is subject to certain broad restrictions. The extension makes use of a particular generalization of the nth roots of unity discovered by Fejér, which he termed a "regularly distributed" point set. The degree of convergence of the interpolating sequence is also investigated, both on and within the curve under consideration. The paper closes with certain extensions of the main theorems to the consideration of simultaneous interpolation to more than one function defined on one or more curves. (Received July 25, 1934.)

264. Mr. E. R. Elliot: *A mixed boundary-value problem for the heat equation.*

We consider a domain D bounded by two characteristics with ordinates h and e, and two arcs $x=X_i(y)$, X_i'' continuous. Segments of the curves $y=h+\delta$, $x=X_i(y)+\lambda_i$ are referred to as δ- and λ_i-displacements. We exhibit the unique function of the form $L(x, y)=\int U(x, y; \xi, h)d\beta(\xi)-2\sum\int U(x, y; X_i(\eta), \eta)dF_i(\eta)$, $U(x, y; \xi, h)=[4\pi(y-h)]\exp[-(x-\xi)^2/(4(y-h))]$, $y>h$, $=0$, $y\leqq h$, where the integration in the first integral is from $X_1(h)$ to $X_2(h)$ and in the second integral from h to y, which satisfies the equation $\partial^2u/\partial x^2-\partial u/\partial x=0$ within D, and satisfies the following boundary conditions: the line integral of $L(x, y)$ along a δ-displacement, as $\delta\rightarrow 0$, approaches $B(x_2)-B(x_1)$, and the line integral of $\partial L(x, y)/\partial x$ along a λ_i-displacement, as $\lambda_i\rightarrow 0$, approaches $G_i(y_2)-G_i(y_1)$, where $B(x)$, $G_i(y)$ are preassigned functions of limited variation, $B(x)$ being continuous at the end points with regular discontinuities elsewhere, $G_i(y)$ continuous at h and continuous from the left elsewhere. The methods employed are also shown to apply to a problem previously discussed by F. A. Dressel (American Journal of Mathematics, vol. 55 (1933), p. 649) and a simpler solution is thereby obtained. (Received July 21, 1934.)

265. Professor O. J. Farrell: *On approximation by polynomials to a function analytic in a simply connected region.*

In a previous paper (this Bulletin, abstract 40-3-152) the writer failed to mention a certain hypothesis to be put upon the region R laid down in the results (1) and (2) of that paper. The description of R should be amended to read as follows: "a finite simply connected region R whose boundary B is such that all the points of the extended plane not on $R+B$ form a single region whose boundary is precisely B." The present paper extends both results (1) and (2) of the earlier paper to an arbitrary limited simply connected region G whose boundary is also the boundary of an infinite region. Neither result can be extended to a region whose boundary is wholly arbitrary. In fact the region G laid down in the present paper is the most general region such that (1) holds for an arbitrary function $f(z)$ analytic and bounded therein. Examples of regions and functions for which (2) does not hold are also given. (Received July 16, 1934.)

266. Professor O. J. Farrell: *On the expansion of harmonic functions in series of harmonic polynomials belonging to a simply connected region.*

Earlier results (this Bulletin, abstract 36-3-71) on the expansion in harmonic polynomials of a function harmonic in an arbitrary finite Jordan region are extended to an arbitrary limited simply connected region whose boundary consists wholly of simple boundary points and is also the boundary of an infinite region. This extension is accomplished through application of certain results by the writer on conformal mapping (American Journal of Mathematics vol. 54 (1932), pp. 571–578). (Received July 16, 1934.)

267. Professor L. R. Ford: *On properties of regions which persist in the subregions bounded by level curves of the Green's function.*

Let the circle $|z|<1$ be mapped conformally on a region S in the w-plane, the origin going into the origin; and let S_r be the map of $|z|<r<1$. Let $\theta(w_1, \cdots, w_n)$ be analytic when $w_1, \cdots, w_n$ range over S and $\theta(0, \cdots, 0)=0$. The following theorem is proved by an application of Schwarz's lemma: *If S has the property that when $w_1, \cdots, w_n$ are in S so also is $w_0=\theta(w_1, \cdots, w_n)$, then S_r likewise has this property.* Special cases are the known theorems that if S is convex so also is S_r [here $\theta=tw_1+(1-t)w_2$]; also if S is star-shaped from the origin so also is S_r. (Received July 30, 1934.)

268. Professor Philip Franklin: *On Minkowski's definition of length and area.*

For convex surfaces in three-space, Minkowski gave a definition of area based on volume. We here show that his method applies to any sufficiently regular surface. It thus gives a simple definition of surface area. (Received July 25, 1934.)

269. Professor Philip Franklin: *A six-color problem.*

For surfaces topologically distinct from the sphere, an argument of Heawood gives a number of colors sufficient for any map. This number is also an upper limit to the number of regions mutually in contact, so that whenever an example of this is at hand, the map coloring problem is solved. Such examples are known for several two-sided surfaces. In this note we show that for the one-sided surface of characteristic zero, while the Heawood number is seven, the maximum number of neighboring regions is in fact six, so that the map coloring problem for this surface is unsolved. (Received July 25, 1934.)

270. Mr. Bernard Friedman: *Sufficient conditions for the solvability of some Pellian equations.*

In the present paper there is discussed the equation $x^2-py^2=B$, where p is a prime. Lagrange's method for solving the equation is supplemented by introducing a further condition on p. This condition on p when combined with the necessary conditions on B and p will be sufficient for the solvability of the above equation. The condition which p will be assumed to satisfy is as follows. For every odd prime $q<p^{1/2}$ of which p is a quadratic residue, integers (x, y) exist satisfying $x^2-py^2=(-1)^{(q-1)/2}q$. Then it is proved that if $B=2^eN$ where N is an odd positive or negative integer and $e=0$ or $\geqq 3$ if $p=8n+1$, $e=0$ or 1 if $p=8n+3$, $8n+7$, $e=0$ or 2 if $p=8n+5$, and if B and p are quadratic residues of each other, there exists a solution in integers of $x^2-py^2=B$. For example, the equation $x^2-67y^2=B$ has integral solutions if and only if B and 67 are quad-

ratic residues of each other and if B does not contain the factor four. (Received July 25, 1934.)

271. Professor J. J. Gergen: *Note on linear operations in functions of bounded variation.*

The object here is to obtain a characterization of linear operators on functions of bounded variation. Let V be the set of functions $x(t)$ which vanish for $t=0$ and are of bounded variation for $0 \leqq t \leqq 1$. Let the operator $L(x)$ be defined over the space V and be linear there, where the norm $\|x\|$ is defined as $\int_0^1 |dx|$. We prove first that there corresponds to each x a function $K_x(t)$ such that (a) $K_x(t)$ is x-measurable and satisfies $|K_x(t)| \leqq$ norm of L for $0 \leqq t \leqq 1$, (b) $L(x) = \int_0^1 K_x(t)dx$, (c) for any two functions x, y, of V, $K_x(t) = K_y(t)$ except possibly on a set $E_{x,y} + E_{y,x}$ where $E_{x,y}$ is of x-measure 0 and $E_{y,x}$ of y-measure 0. As a corollary there follows the result somewhat complementary to F. Riesz's fundamental theorem, that, if V_1 is any separable subspace of V, there exists a $K(t)$ such that $K(t)$ satisfies (a) and $L(x) = \int_0^1 K(t)dx$ for all x in V_1. (Received August 1, 1934.)

272. Professor B. P. Gill: *Zeroless solutions of systems of linear equations in a finite field.*

Let f_i, $(i=1, 2, \cdots, m)$, be homogeneous linear functions of $x_1, x_2, \cdots, x_k$ in the $GF[p^n]$; there is studied in this paper the number σ of solutions in the field of the system of simultaneous equations $f_i=0$ such that no x_j has the value zero. Of the p^{mn} possible linear combinations of the f's with coefficients in the field, let ν_h contain precisely h x's with coefficients not zero. Then $\sigma = p^{-mn} \cdot (p^n-1)^k \sum_{h=0}^{k} \nu_h (1-p^n)^{-h}$. (Received August 1, 1934.)

273. Dr. M. R. Hestenes (National Research Fellow): *The problem of Bolza in the calculus of variations in parametric form.*

In the present paper we give a set of sufficient conditions for a minimum for the problem of Bolza in parametric form. These sufficient conditions have the same generality as those given by the author for the non-parametric case (abstract 40-3-141) in that no normality assumptions are needed. Of particular interest is the treatment of the second variation. We give two methods of studying the second variation. The first is a generalization of a method due to Weierstrass and is particularly adaptable to the present paper. The second method is essentially that of Bliss, and has been used with some modification by Morse, Graves, and others. It consists of introducing a new side condition to the accessory minimum problem. The side condition here used changes the order of anormality of the accessory minimum problem and thus avoids many difficulties which otherwise seem to arise. (Received July 28, 1934.)

274. Professor T. R. Hollcroft: *On nets of algebraic surfaces.*

The jacobian curve J of a net of algebraic surfaces contains all the singularities and contacts of surfaces of the net. The binodes and the pairs of conic nodes of surfaces, and the points of stationary contact and the pairs of points of simple contact of pencils of surfaces of the net lie at definite, but ordinary, points of J. These points of J are located by immersing the net in a web of surfaces. The characteristics of a net of surfaces with basis curves are obtained. The Steinerian curve of a net of surfaces is defined and its projection on a plane is shown to be the reciprocal of the branch-point curve of the transformation established by the (1, 1) correspondence between the surfaces of the net and the lines of a plane. (Received August 1, 1934.)

275. Professor R. E. Langer: *On the asymptotic solutions of ordinary differential equations, with reference to the Stokes phenomenon about a singular point.*

The asymptotic representations of the solutions of a differential equation $u'' + \{\lambda^2\phi(z) + \psi(z)\}u = 0$, for large values of the parameter λ, are known to be subject to the Stokes phenomenon in any domain of the variable which includes a point at which the coefficient $\phi(z)$ becomes zero. The theory and formulas associated with this phenomenon have previously been given for the case in which the coefficient $\psi(z)$ remains bounded. The present paper extends this theory to admit in $\psi(z)$ a pole of the first or second order. It is shown that the presence of such a pole alone engenders the Stokes phenomenon, the latter being then quantitatively determined by the value $(z-z_0)^2\psi(z)$. If $\phi(z)$ and $1/\psi(z)$ vanish together, the phenomenon is determined by both coefficients jointly. The theory here given is applicable to the asymptotic representations of a number of classically important functions, including the ordinary and the associated Legendre functions, the Laguerre polynomials, and the Mathieu functions of higher order. (Received July 26, 1934.)

276. Mr. Norman Levinson: *On the vanishing of a function over an interval.*

Using the Wiener-Paley theorem on the vanishing of a function on the half-line, a condition depending on the behavior of the absolute value of the Fourier transform at infinity is given which is sufficient for the non-vanishing of a function in an interval, no matter how small. This condition is independent of the existence of derivatives, and is hence somewhat different in type from conditions of quasi-analyticity. (Received August 1, 1934.)

277. Mr. Norman Levinson: *A theorem on the magnitude of Fourier transforms.*

In connection with the statement of Wiener that a function and its Fourier transform cannot both be very small at infinity, the theorem will be proved that if $g(u)$ is the Fourier transform of $f(x)$, if $f(x) = O(\exp(-|x|^p(\log|x|)^a))$ and if $g(u) = O(\exp(-|u|^q(\log|u|)^b))$, then if $1/p + 1/q = 1$ and if a and b exceed certain quantitites depending on p and q, $f(x) \equiv g(u) \equiv 0$. In the case $p = \infty$, not included in the above, and already treated by Ingham, a very simple proof of the corresponding theorem may be given, using Wiener's and Paley's theorem on functions vanishing on a half-line. (Received August 1, 1934.)

278. Dr. Hans Lewy: *A priori limitations for solutions of elliptic Monge-Ampère equations.*

Let $x(u, v)$ be an analytic function in $u^2+v^2\leqq 1$, and a solution of a Monge-Ampère equation with analytic coefficients, which is elliptic for the given x. Let M_1, M_2, M_3 be bounds for the absolute values of x, its first, and its second derivatives, respectively. Then there exist bounds for partial derivatives of x of every order, which do not depend on the particular solution x, but merely on the equation and the constants M_1, M_2, M_3. This theorem has important applications in differential geometry in the large. (Received July 31, 1934.)

279. Professor N. H. McCoy: *On certain rings and differential ideals.*

Let K be a commutative ring with unit element in which each ideal has a finite ideal basis. The primary purpose of this paper is to study rings of the form $K[\alpha, \beta]$ where α and β are commutative with elements of K and $\alpha\beta-\beta\alpha$ is commutative with both α and β. A special case of interest is the algebra of quantum mechanics, in which there exist elements p, q such that $pq-qp=1$. A characterization of rings $K[\alpha, \beta]$ is given in terms of the notion of differential ideal in a certain commutative ring. Special attention is given to the case in which K is a field and $K[\alpha, \beta]$ is a finite algebra over K. (Received July 21, 1934.)

280. Dr. Saunders MacLane: *Some unique separation theorems for graphs.*

Since a graph is formed by any number of vertices arbitrarily joined in pairs by edges, it is desirable to have some method of constructing complex graphs from simpler ones. The separation of graphs by subgraphs is one such procedure. Whitney, in the Transactions of this Society, vol. 34 (1932), p. 339, has considered the separation of graphs by single vertices. More generally, any connected graph G is *separated* by a subgraph H if the removal of all the vertices and edges of H disconnects G into two or more pieces. The question of uniqueness at once arises: if a graph is separated as far as possible, is the result independent of the order of the separations? This paper considers this question for separations by chains and by cycles. The successive separation by chains is unique if each chain used is shorter than any other chain joining the same extremities. A similar uniqueness theorem for "shortest" cycles is established. The proofs require investigation of the various symmetrical figures in which two short subgraphs can intersect. Several specimen graphs show that the requirement of "shortness" for unique separation is a natural one. (Received July 30, 1934.)

281. Dr. Saunders MacLane: *A classification of finite distributive lattices.*

The presence of common formal properties for the two dual operations "least common multiple" and "greatest common divisor" suggests the abstract study of all systems or "lattices" having these properties. One natural problem is the investigation of the possible sublattices in a given lattice. For finite distributive lattices it is sufficient to consider two special types of sublattices: the "normal" sublattices and the sublattices which contain the unit element of the original lattice. The normal sublattices are somewhat analogous to normal subgroups; their structure can be easily exhibited in terms of their irreducible elements. Furthermore, these normal sublattices show that any lattice can be split up into two exhaustive and mutually exclusive sublattices. The second type of sublattices, those containing the original unit element, have a more complex structure, but their irreducible elements can be related systematically to the irreducible elements of the original lattice. (Received July 30, 1934.)

282. Professor M. L. MacQueen: *A projective generalization of metrically defined associate surfaces.*

In the metric differential geometry of surfaces in ordinary space, two surfaces are said by Bianchi to be associate if the tangent planes at corresponding points are parallel and if the asymptotic curves on either surface correspond to a conjugate net on the other. The purpose of this paper is to develop a projective generalization of the relation of associateness of surfaces. The first problem is to provide a projectively defined substitute for the metric property of parallelism of surfaces. This provision is made by employing a projective generalization of euclidean parallelism of surfaces developed in the author's thesis and summarized briefly in this paper. One of the well known transformations of surfaces, namely, the fundamental transformation, is employed in constructing a projective analogue of the property of metric parallelism of surfaces. After formulating a definition of projectively associate surfaces, their properties and relations are studied in some detail. A more general type of associateness called modified projective associateness is introduced and is briefly studied. (Received July 15, 1934.)

283. Professor Morris Marden: *On the zeros of the derivative of a rational function.*

In this paper the author redetermines by a new method the locus of the zeros of the derivative of the rational function $f(z)=(z-z_0)^{m_0}(z-z_1)^{m_1}\cdots(z-z_p)^{m_p}$, the m_j being real or complex, when each point z_j varies independently over a given circular region Z_j. He finds again the locus to be bounded by a certain p-circular $2p$-ic curve, which reduces if $\sum m_j=0$ to a $(p-1)$-circular $2(p-1)$-ic curve. The new treatment, which is much shorter and simpler than the one given in the author's doctoral thesis (Transactions of this Society, vol. 32 (1930), pp. 81–109), is based upon the lemma that, if the points u and v independently describe given circular regions, the point $w=u+v$ also describes a circular region. The results also include a determination of the locus of the zeros of any linear combination of $f(z)$ and its derivative. (Received July 31, 1934.)

284. Professor G. A. Miller: *Groups in which the squares of the elements are dihedral groups.*

Various categories of groups can be conveniently studied by means of their subgroups composed of the elements which are squares of other elements of the group. In the present article the author considers the category in which these subgroups are dihedral. There is only one abelian dihedral group and this is the case which presents the greatest difficulties as regards the category of groups under consideration. The paper has been accepted for publication in the Transactions of the Society. (Received July 23, 1934.)

285. Dr. E. P. Northrop: *An operational solution of the Maxwell field equations.*

This paper is concerned with the solution of the Maxwell electrodynamic field equations in the case of the simplest type of field, that is, free space, containing no charges or currents. By a proper choice of units, these equations can be expressed in the form $\nabla\cdot\boldsymbol{E}=0$, $\nabla\cdot\boldsymbol{H}=0$, $\nabla\times\boldsymbol{E}=-\partial\boldsymbol{H}/\partial t$, $\nabla\times\boldsymbol{H}=\partial\boldsymbol{E}/\partial t$, where $\boldsymbol{E}$ and $\boldsymbol{H}$ are respectively the electric and magnetic intensities. If we denote by $\boldsymbol{V}$ the six-component vector $(E_x, E_y, E_z, H_x, H_y, H_z)$, the two curl equations can be written as the single equation $\partial\boldsymbol{V}/\partial t=iH\boldsymbol{V}$, where H is a certain matrix differential operator. The solution of this equation is formally $\boldsymbol{V}=e^{iHt}\boldsymbol{V}_0$ where $\boldsymbol{V}_0$ characterizes the initial state of the field. The operator e^{iHt} is determined in a rigorous manner by means of the operational calculus, as developed in M. H. Stone's treatise *Linear Transformations in Hilbert Space*. The significance of the divergence conditions is briefly discussed. The work results in a general integral representation for the vector $\boldsymbol{V}$ characterizing the electromagnetic field at any point and at any time. (Received July 19, 1934.)

286. Mr. L. B. Robinson: *A singular solution of a functional equation.*

Consider a special Izumi equation (1) $u'(x)=P_n(x)u(x/2)+Q_n(x)$. Where P_n and Q_n are polynomials of degree n. Write $u(x)\equiv v(1/x)\equiv v(\xi/2)$. The above equation becomes (2) $v'(\xi/2)\xi^{n+2}=R_n(\xi)v(\xi)+S_n(\xi)$, $R_n(0)\neq 0$. Equation (1) has a solution everywhere holomorphic. Equation (2) has a solution holomorphic so long as $R_n\neq 0$. In the neighborhood of $x=0$, $\xi=\infty$, v is uniform or non-uniform. If uniform, anessential singularity exists. Then by the Picard-Weierstrass theorem on essential singularities, it cannot be identified with u. If v is non-uniform, it cannot, ipso facto, be identified with u. So at last an example of a singular solution has been found. To avoid using the law of the excluded middle, the author by resting on Liouville's theorem has proved the same result. (Received July 14, 1934.)

287. Mr. R. Q. Seale: *A simple proof of Minkowski's theorem concerning non-homogeneous linear forms.*

In this paper a new proof is given of Minkowski's theorem that if $\Delta=A_1B_2-A_2B_1=1$, integers x, y always exist such that $|(A_1x+B_1y+C_1)(A_2x+B_2y+C_2)|\leqq\frac{1}{4}$; it is further proved that if A_1/B_1 is irrational, $|A_1x+B_1y+C_1|$ can, at the same time, be made arbitrarily small. The proof is simple, direct, and is based on no ideas more advanced than the elementary properties of convergents. (Received July 30, 1934.)

288. Professor I. M. Sheffer: *A local solution of the difference equation $\Delta y=f$ and of related equations.*

The equation $\Delta y=f$ has been studied from a number of viewpoints. The more important are (1) with respect to the asymptotic character of $f(x)$ at infinity; (2) concerning proofs that if $f(x)$ is an entire function (meromorphic function) then an entire (meromorphic) solution exists. So far as we are aware, however, there has been no treatment for an arbitrary analytic function. The present paper aims to fill this gap. We obtain a local existence theorem. The method consists in the application of the Borel integral and its inverse. The same method, leading to the same conclusions, applies to related equations such as $\sum_1^n \alpha_i y(x+c_i)=f$, $\sum_1^n \alpha_i y^{(i)}(x)=f$, and also to equations involving certain more general differential operators with constant coefficients. (Received July 28, 1934.)

289. Dr. Mildred M. Sullivan (National Research Fellow): *On the partial derivatives of harmonic functions as functionals of the determining boundaries.*

In this paper, the partial derivatives of harmonic functions determined in regions of space by their values on the boundaries of the regions are considered as functionals of the boundaries, and conditions for the continuity of these functionals with respect to the boundaries are obtained. (Received July 30, 1934.)

290. Dr. Henrietta Terry: *Abelian subgroups of the I-group of the abelian group of order p^n and type* 1, 1, $\cdots$.

This paper determines a set of maximal abelian subgroups of a Sylow subgroup I_p of I in which every abelian subgroup of I_p is contained. Methods are developed for the classification of subgroups of order p^2 of I_p, and the classification is carried out for some of the earlier cases. (Received July 13, 1934.)

291. Professor J. V. Uspensky: *On the expansion of the remainder in the Newton-Cotes formula.*

The author shows that the remainder of the Newton-Cotes quadrature formula can be expanded into series quite similar to the classical Euler-Maclaurin series. The general proof in this case is more difficult than in the case of the Gaussian formula and requires an elaborate study of some properties of Cotes coefficients. (Received July 30, 1934.)

292. Professor Louis Weisner: *Irreducibility of polynomials of degree n which assume the same value n times.*

The polynomial $f(x)=ax(x-t_1)\cdots(x-t_{n-1})\pm k$, in which a, k, $t_1, \cdots, t_{n-1}$ are positive integers and the t's distinct, is proved to be irreducible in the field of rational numbers if at least one of the n inequalities $a>2^nk^2$, $t_i>4nk$, $(i=1, \cdots, n-1)$, is satisfied. Every polynomial of degree n which assumes the same value for n distinct integral values of the variable is equivalent, under simple transformations which do not affect the reducibility of the polynomial, to one of the form of $f(x)$. Hence only a finite number of non-equivalent reducible polynomials of degree n exist which assume a given integral value for n different integral values of the variable. (Received July 27, 1934.)

293. Dr. Hassler Whitney: *Analytic approximations to manifolds.*

Let M be a point set in n-space E_n. We shall say M is an *m-manifold in regular position* in E_n if the following is true: (1) There are $n-m$ continuous vector functions $f_i(p)$ defined over M, such that $f_1(p), \cdots, f_{n-m}(p)$ are linearly independent for each p. (2) For each p in M there is a neighborhood U of p and an $\epsilon>0$ such that (a) U is an m-cell, (b) if p_1 and p_2 are in U, then the line through p_1 and p_2 makes an angle $\geqq \epsilon$ with the $(n-m)$-plane through p_1 determined by $f_1(p_1), \cdots, f_{n-m}(p_1)$. Certain classes of manifolds are shown to have this property. Theorem: *Given M as above and a continuous positive function $\phi(p)$ defined over M, there is an analytic manifold M' homeomorphic with M, such that corresponding points p and p' of M and M' are within a distance $\phi(p)$ of each other.* (Received July 23, 1934.)

294. Dr. Hassler Whitney: *On the abstract properties of linear dependence.*

Let M be a set of elements $C_1, \cdots, C_n$, and let any subset either be or be not "(linearly) independent." Suppose (1) any subset of an independent set is independent, and (2) if $C_1, \cdots, C_m$ and $C_1', \cdots, C_{m+1}'$ are independent, then for some i, $C_1, \cdots, C_m, C_i'$ are independent. Then we shall call M a *matroid.* Examples of sets of elements forming a matroid are the columns of a matrix, and the arcs of a linear graph (a set of arcs being independent if it contains no circuit). Other sets of axioms for matroids are given, in terms of bases, circuits, rank. Properties of matroids (especially of non-separable and dual matroids) are found corresponding to theorems on graphs in a paper by the author (Transactions of this Society, 1932). Dual matrices have a simple geometrical interpretation. Matroids form a larger class than matrices, as an example of a matroid with nine elements shows. (Received July 23, 1934.)

295. Mr. L. B. Robinson: *The equation of Izumi and a contour problem.*

Izumi (Tôhoku Mathematical Journal, 1929) has considered the equation (1) $u'(x)-\phi \equiv u'(x)-a(x)u(\omega(x))-c(x)=0$. He demonstrates the existence of a solution in the region $|x|<\rho$, $|\omega(x)|<\rho$. His method seems to fail when we consider the case (2) $u'(x)-f=u'(x)-a(x)u(\omega(x))-b(x)u(x)-c(x)=0$ under the hypothesis that when $|x|\leqq 1$, then $a(x)$, $b(x)$, $c(x)$, and $\omega(x)$ are regular and also $|\omega(x)|\leqq 1$. But the author replaces (2) by its equivalent $u'(x)-f=0, \cdots, u^n(x)-[f^{(n-1)}]=0$. The brackets signify that all derivatives which appear in the f have been eliminated. The author obtains a solution which is valid in a domain D and on its *contour.* He can extend his results to the case $(a\partial/\partial x)^n u(x)-f=0$. (Received August 2, 1934.)

296. Professor P. A. Smith: *The fundamental groups of group manifolds.*

The principal result of this paper is that the maximum number of linearly independent elements in the (abelian) fundamental group of an r-parameter group manifold is at most r. (Received August 4, 1934.)

297. Professor Otto Szász: *Convergence properties of Fourier series.*

Let (B) be the class of series $\sum c_n$ such that the series $\sum (c_n - |c_n|)$ is "slowly oscillating below." The present paper deals with various convergence and boundedness properties (uniform convergence and uniform boundedness) of Fourier cosine and sine series whose coefficients belong to the class (B). The results of the paper can be extended to almost periodic functions and to Fourier integrals. (Received August 6, 1934.)

298. Professor W. D. Baten: *A formula for finding the skewness of the distribution made up of a finite number of samples.*

This article develops a formula for the skewness of a frequency distribution made up of the variates in a finite number of samples in terms of the numbers of variates in each sample, the means, standard deviations, and skewness of the various samples. This sum formula is obtained by expressing the characteristics of each sample in terms of the fundamental summations of the variates. At the close of the article there is an illustration indicating the use of the formula and an effective way of carrying out the computations. (Received August 10, 1934.)

299. Mr. Garrett Birkhoff: *Integration of functions with values in a Banach space.*

The point sets of any vector space constitute a "vectoroid space" in which only the postulate $x+(-x)=0$ fails. Relative to this space, convex hulls define a homeomorphism. Based on these notions, and the "unconditional" or commutative convergence of Orlicz, a process of integration for functions with values in a Banach space is created which is effectively more powerful than the method of Bochner. As usual, definite integrals are additive; but the space of integrable functions is not in general complete. If the Banach space is of finite dimensions, our method is in effect Lebesgue's. (Received August 4, 1934.)

300. Dr. L. A. Dye: *The number of trisecants of a space curve of order m which meet an i-fold secant.*

By means of a correspondence, the number of trisecants of a space curve of order m which meet an i-fold secant is determined to be $(m-2)[h-m(m-1)/6] -i(h-m+2)+i(i-1)(i-2)/6$. (Received August 15, 1934.)

301. Professor Marston Morse and Mr. G. B. Van Schaack: *The boundary conditions in the critical point theory.*

Morse has previously treated the problem of the critical points of a real analytic function f of n variables with no restrictions on the nature of the critical points or the function on the boundary. However, the function f was required to have a positive normal derivative on the boundary. In the present paper this requirement is replaced by the condition that f have no critical points on the boundary. The results are obtained under hypotheses which include cases broader than the analytic case. (Received August 7, 1934.)

302. Professor Marston Morse and Mr. G. B. Van Schaack: *The critical point theory under general boundary conditions.*

The present paper has two purposes. The first is to simplify as far as possible the treatment of critical points of non-degenerate functions of n variables. The second is to extend the earlier treatment of Morse to the case of general boundary conditions. The boundary condition that the normal derivative f_N of f be positive on the boundary is replaced in the general boundary conditions by the requirement that f have no critical points on the boundary, the boundary function ψ being non-degenerate. The function f is replaced by a function F which is identical with f except neighboring the boundary and which has a normal derivative F_N positive on the boundary. The new function F has as additional critical points a critical point of index k neighboring each critical point of index k of ψ at which $f_N < 0$. (Received August 7, 1934.)

303. Professor Marston Morse: *Sufficient conditions in the problem of Lagrange without assumptions of normalcy.*

Sufficient conditions in the ordinary Lagrange problem in a form which involves the Jacobi or conjugate point condition have not previously been established without assumptions of normalcy, although considerable effort has been made to reduce such assumptions. The present paper establishes such conditions for the first time. (Received August 7, 1934.)

304. Professor G. C. Evans: *Short proof of Kellogg's lemma.*

Let T be an unbounded domain whose frontier t is a closed bounded, reduced set; let the complement of T be CT; let a conductor potential of t be $V(M)$. A short proof of Kellogg's lemma, that t contains at least one point regular with respect to T, is contained in the following statements: (a) A sufficient condition that Q be regular is that $V(Q)$ have the value 1 (Vasilesco). (b) The upper bound of $V(M)$ is 1 in any neighborhood of any point of t (Kellogg, Vasilesco, Bouligand). (c) If $V(M)$ is continuous at Q of t, for M on t, it is then continuous at Q for arbitrary approach of M (Evans). (d) $V(M)$ is of Baire class 1 in CT, and is therefore punctually discontinuous (Baire), so that there are points Q of t where $V(M)$ is continuous for M on t. (Received August 17, 1934.)

305. Dr. Hillel Poritsky: *On the nature of the "reflection" in a plane boundary corresponding to the boundary condition* $\partial u/\partial n + \text{const.}\ u=0$.

The "reflection" or analytic continuation of a harmonic function u across a plane boundary along which either of the conditions $u=0$, $\partial u/\partial n=0$ holds, is well known. In this note the reflection across a plane boundary $x=0$ along which the boundary condition $\partial u/\partial x+au=0$, $a=\text{const.}\neq 0$, holds, is investigated. It is shown that the Green's function for this condition, with a pole at P in $x>0$, when continued analytically to $x<0$ has the following singularities: (a) a positive image at P', the reflection of P in $x=0$; (b) an exponential trail of negative images along the line through P_1P', from P' to $x=-\infty$. Similar "reflections" are established for the heat conduction and certain other partial differential equations. (Received September 5, 1934.)

306. Dr. W. T. Reid: *The theory of the second variation for the non-parametric problem of Bolza.*

The principal result obtained in this paper is as follows: If E_{12} is an extremal arc for a non-parametric problem of Bolza having multipliers $\lambda_0=\text{const.}$, $\lambda_\alpha(x)$ with which it satisfies the strengthened Clebsch condition, and there is no point on E_{12} conjugate to the point 1, then there exists a family of mutually conjugate solutions of the accessory equations whose determinant is non-zero along E_{12}. In view of this result, the sufficiency theorems of Morse (Annals of Mathematics, vol. 33 (1932), pp. 261–274) for an extremal E_{12} which is normal on x_1x_2 are valid without the assumption of normality on sub-intervals x_1x_3 $(x_1<x_3<x_2)$ that has been used by each of these authors. Similarly, the sufficiency theorems of Bliss and Hestenes (Transactions of this Society, vol. 35 (1933), pp. 305–326 and pp. 479–490) for the problem of Mayer are valid with correspondingly weakened hypotheses. Hestenes (abstracts 39–5–124, 39–7–213, and 40–3–141) has obtained sufficiency theorems in the problem of Bolza assuming no normality conditionsa side from the existence of multipliers of the form $\lambda_0=1$, $\lambda_\alpha(x)$, by modifying the usual form of the Mayer condition. The results of this paper enable one to state these theorems with the Mayer condition phrased in either the form used by Morse or that used by Bliss. (Received September 5, 1934).

307. Mr. W. H. Ingram: *Dynamics of the homopolar generator.*

The slip-rings are more than a means to avoid twisting conductors in a Maxwell net, and constitute dynamically essential elements in the machine. The voltage at the brushes is given by Lagrangean multipliers, and in steady running with constant field current I the internal e.m.f. for the axial type is given by $(4/\pi)Im(d\theta/dt)$ where m is the inductance mutual to field circuit and slip-ring. The physical coordinates are true, as in all slip-ring machines, and the space of the trajectories, as for all machines of any kind, is Riemannian. Null-functions of the form $\lambda_i(\sum A_{ij}(dq_j/dt))$ are added to the activity function in a transformation of variable to yield the transformed Lagrangean multipliers. (Received September 7, 1934).

308. Professor E. T. Bell: *Ternary arithmetical identities.*

The addition theorems for the elliptic theta functions imply a set of precisely four identities, none of which can be derived from others by transformations of the first and second orders, which are bilinear in theta functions and doubly periodic functions of the second kind. The arithmetical equivalents of this set refer to all representations of positive integers in the form x^2+yz, with y, z non-negative; the arithmetical function involved is $L(u, v, w)$, which is finite and single-valued for integer values of u, v, w, and is an alternating function of u, v, w. One of the four is equivalent to the fundamental identity of Uspensky; the other three are new. The set exhausts the identities of this type implied by the addition theorems for the thetas. (Received August 31, 1934.)

309. Professor A. F. Moursund: *On summation of derived series of the conjugate Fourier series*. II.

This paper is an extension of the author's paper *On summation of derived series of the conjugate Fourier series* (abstract 39-9-256, this Bulletin). Three theorems concerning the N_{z_p} summability of the rth derived series of the conjugate Fourier series are given. These theorems and a theorem of the earlier paper yield, by specialization of the N_{z_p} method, theorems for the Bosanquet-Linfoot and Cesàro methods. The case $r=0$ gives four theorems, three well known and one new, for the Cesàro summability of the conjugate Fourier series. (Received August 7, 1934.)

ERRATUM

Volume 40, page 388, abstract No. 204 (by Sister Mary Cleophas Garvin, S.N.D.): the formula for the series in the first line, which was printed as $\sum_{n=1}^{\infty} a_n z^{\lambda n}/(1-z^{\mu n})$, should read $\sum_{n=1}^{\infty} a_n z^{\lambda_n}/(1-z^{\mu_n})$.