CLASSES OF MAXIMUM NUMBERS ASSOCIATED
WITH CERTAIN SYMMETRIC EQUATIONS
IN n RECIPROCALS*

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1. Introduction. In an article dealing with this subject Sim-
monsf stated without proof two general theorems whose proofs
are to be obtained by making certain modifications in the theory
of his article, which will be referred to in the sequel as I (for
paper I). These theorems will be stated after a few definitions
from I are recalled.

Kellogg solution. If a solution \( x = (x_1, x_2, \ldots, x_n) \) of any given
symmetric equation in \( n \) reciprocals is obtained by minimizing
the variables \( x_1, x_2, \ldots, x_{n-1} \) (all positive integers) in this
order, one at a time, we shall denote it by \( w \) and call it the Kel-
logg solution of the given equation. Thus \( x = (2, 3, 6) \) is the Kel-
logg solution \( w \) of the equation \( x_1^{-1} + x_2^{-1} + x_5^{-1} = 1 \).

E-solution. A solution \( x = (x_1, x_2, \ldots, x_n) \) of a given sym-
metric equation in \( n \) reciprocals is called an E-solution if \( x_1, \)
\( x_2, \ldots, x_{n-1} \) are positive integers and \( x_1 \leq x_2 \leq \cdots \leq x_n \).

Polynomial \( P(x) \). Let \( P(x_1, x_2, \ldots, x_n) = P(x) \) be any poly-
nomial which is symmetric in the \( n \) variables \( x_i \), contains one
or more positive coefficients and no negative coefficient, and is
not identically a constant.

\( \Sigma_{i,j}(x) \). With \( i \geq 0 \) and \( j \) equal to integers, we let \( \Sigma_{i,j}(x) \)
stand for the \( j \)th elementary symmetric function of the \( i \) vari-
ables \( x_1, x_2, \ldots, x_i \); with the customary understanding that

\[
\Sigma_{i,j}(x) \begin{cases} 
0 & \text{when } i < j \text{ and also when } j < 0, \\
1 & \text{when } j = 0.
\end{cases}
\]

We now state the two theorems referred to above with the
numbering of I.

Theorem 4. If in the equation

* Presented to the Society, April 6, 1934.
† See H. A. Simmons, Transactions of this Society, vol. 34 (1932), pp. 876–
907.
\[ \sum_{n,r}(1/x) + \lambda_{r+1}\sum_{n,r+1}(1/x) + \lambda_{r+2}\sum_{n,r+2}(1/x) + \cdots + \lambda\sum_{n,s}(1/x) = b/a, \quad a \equiv [(c+1)b-1], \]

where \( b \) and \( c \) are positive integers, either \( \lambda_p = 1, (p=r+1, \cdots, s) \), or \( \lambda_{r+1} \) is an integer \( \geq 0 \) and \( \lambda_p = 0, (p=r+2, \cdots, s) \), the largest number that exists in any \( E \)-solution of the resulting equation (21) is the \( w_n \) of the corresponding solution \( w \) defined by the recurrence relations

\[ w_p = 1, \quad (p = 1, \cdots, r - 1), \]
\[ w_{p+1} = a[\sum_{n,p-r+1}(w) + \lambda_{p+1}\sum_{p,p-r}(w)] + 1, \quad (p = r, \cdots, n - 2), \]
\[ w_n = a[\sum_{n-1,n}(w) + \lambda_{n-1,n-1}\sum_{n-1,n-r-1}(w) + \lambda_{n-2}\sum_{n-1,n-r-2}(w) + \cdots + \lambda\sum_{n-1,n-s}(w)]. \]

Furthermore, in each of these cases, \( w_n \) appears in but one \( E \)-solution of the equation (21) in question.

**Theorem 5.** In each of the two cases of Theorem 4, if \( X \) is an \( E \)-solution of equation (21) and is different from the \( w \) of that equation, then \( P(X) < P(w) \).

Here we shall extend all theorems of I that relate to maximum numbers by considering cases in which the right members, \( b/a \), of the equations in question are much more general than they were in I. For (a very simple) example, consider

\[ \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{3}{3c-2}, \]

where \( c \geq 3 \) is odd. The Kellogg solution of this equation is readily found to be \( x = w \), where \( w_1 = c, w_2 = 2^{-1}(3c^2-2c+1), w_3 = (3c-2)w_2w_3 \). We show later that the theorems above hold for this solution \( w \). This conclusion does not follow from I because \( (3/(3c-2)) - (1/w_1) \) is not a unit fraction, while in I, \( (b/a) - (1/(w_1 \cdots w_n)), (r=1, \cdots, n-1) \), was required to be a unit fraction. Our new theorems are stated in §§6, 9, 10.

We divide the portion of this paper from §2 to the end into two parts. In Part 1, §§2 to 5, we prove a case of Theorem 4; in Part 2, we develop our extensions as far as seems desirable.

Except where the contrary is stated, we employ here the definitions and notation of I. The analogs of equation (i) and
Theorem \( i \) of I, if written here, are denoted by \((ia)\) and Theorem \( \bar{ia} \), respectively, and proofs that are to be obtained as were analogous proofs of I are omitted. Consequently, in Part 1 we only state our modifications of certain definitions and lemmas and prove inequality \((46a)\).

**PART 1. PROOF OF ONE CASE OF THEOREM 4**

2. *New Relations.* We shall prove the part of Theorem 4, \( \S 23 \) of I, for which \( \lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_{s}=1 \), that is, the case where equation \((21)\) is

\[
\phi_n(1/x) = b/a,
\]

where \( \phi_n(1/x) = \Sigma_{n,r}(1/x) + \Sigma_{n,r+1}(1/x) + \cdots + \Sigma_{n,s}(1/x) \), and \( r, s, n \) are positive integers such that \( r < s \leq n \).

If in the last displayed equation \( n \) is replaced by \( p, r \leq p \leq n \), our definition of \( \phi_p(1/x) \) is obtained.

The analogs here of \((26), (28), (29), (30), (32)\) of I are as follows.

\[
w_p = 1, \quad (p = 1, \cdots, r - 1), \quad w_r = c + 1,
\]

\[
w_{p+1} = a[\Sigma_{p,p-r+1}(w) + \Sigma_{p,p-r}(w) + \cdots
\]
\[\quad + \Sigma_{p,p-s+1}(w)] + 1, \quad (p = r, \cdots, n - 2),
\]

\[
w_n = a[\Sigma_{n-1,n-r}(w) + \Sigma_{n-1,n-r-1}(w) + \cdots
\]
\[\quad + \Sigma_{n-1,n-s}(w)].
\]

\[
\phi_p(1/w) = (bw_1 \cdots w_p - 1)/(aw_1 \cdots w_p), \quad (p = r, \cdots, n - 1; \text{see (27) of I}).
\]

\[
\phi_p(1/X) \leq (bX_1 \cdots X_p - 1)/(aX_1 \cdots X_p), \quad (p = r, \cdots, n - 1),
\]

which is to be proved as was \((29)\) of I.

\[
\phi_{n-1}(1/X) \leq \phi_{n-1}(1/w), \quad (1 \leq r < s \leq n).
\]

\[
\Sigma_{n-1,r-1}(1/X) + \Sigma_{n-1,r}(1/X) + \cdots + \Sigma_{n-1,s-1}(1/X)
\]
\[
\leq \Sigma_{n-1,r-1}(1/w) + \Sigma_{n-1,r}(1/w) + \cdots + \Sigma_{n-1,s-1}(1/w), \quad (1 \leq r < s \leq n).
\]
3. New Definitions of Sets \( \sigma, \tau, \) and of the Transformation.

Sets* \( \sigma, \tau. \) Let \( \lambda \) be a fixed positive integer such that \( r \leq \lambda \leq n, \) where \( r \) and \( n \) are as they are defined for (21a). We shall call \( x_1 \ldots x_\lambda \) a set \( \sigma \) (relative to the \( w \) of (26a)) if, and only if, \( \phi_p(1/x) \leq \phi_p(1/w) \) for every positive integer \( p \) such that \( r \leq p \leq \lambda. \) We shall call \( x_1 \ldots x_{(\lambda+1)} \) a set \( \tau \) if, and only if, with \( \lambda \) a positive integer such that \( r \leq \lambda \leq n-2, \) \( x_1 \ldots x_{(\lambda+1)} \) is not, and \( x_1 \ldots \lambda \) is, a set \( \sigma. \)

The new transformation.† Suppose \( X_1 \ldots r \) contains at least one element of each of the classes \( A \) and \( B \) defined in I. Then we define our transformation of \( X_1 \ldots r \) into a new set \( X_1' \ldots r' \) by (t\(_{2a}\)) or (t\(_{4a}\)):

\[
(t_{2a}) \quad X_p' = X_p(p \neq q_1, q, p \leq \nu), \quad X_{q_1} = w_{q_1},
\]

\[
\phi_p(1/X') = \phi_p(1/X);
\]

\[
(t_{4a}) \quad X_p' = X_p(p \neq q_1, q, p \leq \nu), \quad X_{q_1} = w_{q_1},
\]

\[
\phi_p(1/X') = \phi_p(1/X),
\]

according as (t\(_{2a}\)) requires \( X_{q_1} \) to be not greater than \( w_{q_1} \) or greater than \( w_{q_1} \), respectively.

4. Analogs of Lemmas 6, 7, 8.

**Lemma 6a.** (i) If \( X_1 \ldots k \) is a set \( \tau, \) \( X_1 \ldots k \) is transformable.‡

(ii) If \( X_1 \ldots k \) is a set \( \tau \) or a transformable set \( \sigma \) for which \( r < k \leq n, \) and if \( i \) is a positive integer, application of (33a) with \( \nu = k \) to \( X_1 \ldots k \) yields a set \( X_1' \ldots k' \) such that (34a), identical with (34), is true, and

\[
(35a) \quad \phi_p(1/X') \leq \phi_p(1/w), \text{ for } p = r, \ldots, k - 1;
\]

[(36a) is not needed];

\[
(37a) \quad \phi_p(1/X') < \phi_p(1/X), \text{ for } p = q_1, \ldots, k - 1;
\]

\[
(38a) \quad \phi_p(1/X') = \phi_p(1/X), \text{ for } p = k.
\]

**Lemma 7a.** If \( s_1, s_2, \lambda \) are integers \( \geq 0 \) and \( s_1 \geq s_2, \lambda > 0, \) then

\[
(1^{s_1})^{(1^{s_2})} > (1^{s_1})^{(1^{s_2})}, \lambda > 0.
\]

* See pp. 889–890 of I.
† See p. 891 of I.
‡ See p. 891 of I.
In proving the next lemma by our method, Lemma 7a would be used.

**Lemma 8a.** If \( u, v, \gamma, t \) are integers with \( u > v \geq \gamma \geq 1, 0 \leq t \leq (\gamma - 1) \), and \( x_i > 0 \) for \( i = 1, \ldots, u \), then

\[
\Sigma_u,\gamma(1/x)\Sigma_v,\gamma(1/x) > \Sigma_u,\gamma(1/x)\Sigma_v,\gamma(1/x).
\]

5. *Analog of Relation (46) of I.* From (37a) and (38a) one may obtain (46a) below as (46) was obtained from (37) and (38) of I:

\[
(46a)* \quad \sum_{k-2,s-1}^{r-2} + \sum_{k-2,s}^{r-1} + \cdots + \sum_{k-2,s-2}^{r-2} \geq \sum_{k-2,s-1}^{r-2} + \sum_{k-2,s}^{r-1} + \cdots + \sum_{k-2,s-2}^{r-2}.
\]

With

\[
c_1 = \sum_{k-2,s-1}^{r-2} + \sum_{k-2,s}^{r-1} + \cdots + \sum_{k-2,s-2}^{r-2},
\]

\[
c_2 = \sum_{k-2,s-1}^{r-2} + \sum_{k-2,s}^{r-1} + \cdots + \sum_{k-2,s-2}^{r-2},
\]

(46a) may be written

\[
(\sum_{k-2,s-1}^{r-2} + c_1)(\sum_{k-2,s-1}^{r-2} + c_2) > (\sum_{k-2,s-1}^{r-2} + c_1)(\sum_{k-2,s-1}^{r-2} + c_2).
\]

**Proof of (46a).** That this inequality is true follows from the inequalities (I, I, I) below, which we presently prove. In arriving at (46a), one finds that it is to hold under the hypotheses \( 1 \leq r \leq 1q \leq p < k \), with \( 1q \geq 2 \). These relations are understood to hold in this proof.

\[
(I_1) \quad c_1 \Sigma_{k-2,s-1}^{r-2} > c_2 \Sigma_{k-2,s-2}^{r-2},
\]

\[
(I_2) \quad c_2 \Sigma_{k-2,s-1}^{r-2} \geq c_1 \Sigma_{k-2,s-1}^{r-2},
\]

\[
(I_3) \quad \Sigma_{k-2,s-1}^{r-2} \Sigma_{k-2,s-1}^{r-2} \geq \Sigma_{k-2,s-1}^{r-2} \Sigma_{p-2,s-1}^{r-2}.
\]

The terms in \( c_1 \) are \( \sum_{k-2,s-1}^{r-2} (\lambda = r-1, \ldots, s-2) \); those in \( c_2, \sum_{p-2,s-1}^{r-2} \). Hence (I, I, I) may be written

\[
(I') \quad \Sigma_{k-2,s-1}^{r-2} \Sigma_{p-2,s-1}^{r-2} > \Sigma_{k-2,s-1}^{r-2} \Sigma_{p-2,s-1}^{r-2},
\]

\[
(I'_*) \quad \Sigma_{k-2,s-1}^{r-2} \Sigma_{p-2,s-1}^{r-2} \geq \Sigma_{k-2,s-1}^{r-2} \Sigma_{p-2,s-1}^{r-2}.
\]

* \( \Sigma_{k-2,s-1}^{r-2} = \Sigma_{k-2,s-1}^{r-2}(1/x) \).
respectively. If \( p - 2 \geq \lambda \), \((I'')\) holds by Lemma 8a with \( u = k - 2 \), \( v = p - 2 \), \( \gamma = \lambda \), and \( t = r - 2 \). If \( (p - 2) < \lambda \), then since \( p \geq r \) the left side of \((I'')\) is positive and the right side is zero, so that \((I'')\) holds. Consider \((I'_2)\). If \( (p - 2) \geq (s - 1) \), \((I'_2)\) holds with \( > \) by Lemma 8a, with \( u = k - 2 \), \( v = p - 2 \), \( \gamma = s - 1 \), \( t = \lambda \). If \( (p - 2) < (s - 1) \), the right side of \((I'_2)\) is zero and the left side is surely not negative. Therefore \((I'_2)\) holds with \( \geq \). The proof for \((I_2)\) is a special case of that for \((I'_2)\).

**PART 2. EXTENSIONS OF PREVIOUS THEORY**

6. A New Theorem on Equations of the Form \( \Sigma_{n,1}(1/x) = b/a \).

The article I contains results for equations of this type only when \( a = [(c+1)b-1] \) and \( b, c \) are positive integers. This case is covered by taking \( \mu = 1 \) in the new Theorem A below.

**THEOREM A.** Suppose \( a, b \) and \( \mu, \nu \), \( \mu \leq (n - 1) \), are given positive integers, with \( a \) and \( b, b \leq a, \) relatively prime, and that there exists a set of \( n \) numbers \( w = (w_1, w_2, \ldots, w_n) \) with the following properties:

1. it is an E-solution of the equation \( \Sigma_{n,1}(1/x) = b/a \);

2. \[
\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{b w_1 \cdots w_p - 1^*}{a w_1 \cdots w_p}
\]

for every positive integral value of \( p \) for which \( \mu \leq p \leq (n - 1) \) and for no smaller value of \( p \);

3. if \( x = X \), where \( X \neq w \), is an E-solution of the equation in 1°, then for every positive integral value of \( p \leq \mu \) for which \( X_1 \cdots p \neq w_1 \cdots p \), the inequality \( \Sigma_{p,1}(1/X) < \Sigma_{p,1}(1/w) \) holds.

For such a set \( w \) the following conclusions hold:

(i) \( w_n \) is the largest number that exists in any E-solution of the equation in 1° and \( w_n \) appears in no E-solution of this equation except \( w \);

(ii) \( P(x) < P(w). \)

**Proof of (i).** It follows from hypotheses \( (j^o), (j=1, 2, 3) \), that the \( w \) described in Theorem A is the Kellogg solution of the equation in 1°. This fact will be used in the rest of the proof of (i), which we presently make by considering separately the following cases: \( (\alpha) \mu = n - 1; \ (\beta) \mu = n - 2; \ (\gamma) \mu \leq (n - 3) \).

* See equation (28) of I, with \( r = 1 \).
† See p. 887 of I.
(α) In this trivial case, (3°) requires that conclusion (i) be true.

(β) If \( X_1 \ldots \mu = w_1 \ldots \mu \), since \( X \neq w \) is an \( E \)-solution of the equation in (1°) while \( w \) is its Kellogg solution, \( X_{\mu+1} = X_n-1 > w_{n-1} \), and (i) is true. If \( X_1 \ldots \mu \neq w_1 \ldots \mu \), we reach the desired conclusion as follows. From the hypotheses (j°), \( j = 1, 2, 3 \), and the assumption that \( X_1 \ldots \mu \neq w_1 \ldots \mu \), with \( \mu = n-2 \), we have

\[
\Sigma_{p,1}(1/X) \leq \Sigma_{p,1}(1/w), \quad (p = 1, \ldots, \mu - 1);
\]

(1) \[ \Sigma_{n-2,1}(1/X) < \Sigma_{n-2,1}(1/w) = \frac{b w_1 \ldots w_{n-2} - 1}{a w_1 \ldots w_{n-2}}; \]

(2) \[ \Sigma_{n-1,1}(1/X) \leq \frac{b X_1 \ldots X_{n-1} - 1}{a X_1 \ldots X_{n-1}}, \quad \text{[see (29) of I]}; \]

and we wish to contradict the assumption which we now make, namely that

(3) \[ \Sigma_{n-1,1}(1/X) \geq \Sigma_{n-1,1}(1/w) = \frac{b w_1 \ldots w_{n-1} - 1}{a w_1 \ldots w_{n-1}}. \]

Relations (2) and (3) imply that

(4) \[ X_1 \ldots X_{n-1} \geq w_1 \ldots w_{n-1}; \]

(1) and (3), that \( X_1 \ldots (n-1) \) is transformable. Since \( X \) is an \( E \)-solution, it follows now that when exhaustive applications of (33a) for \( X_1 \ldots (n-1) \) are made, the following relations hold [see (53) of I]:

(5) \[ X_1 \ldots X_{n-1} < X_1' \ldots X_{n-1}' \leq w_1 \ldots w_{n-1}. \]

Since (4) and (5) are contradictory, we conclude that (3) is false. Consequently conclusion (i) holds in case (β).

(γ) The demonstration that is required here is essentially the same as that given under (β) for the case \( X_1 \ldots \mu \neq w_1 \ldots \mu \); induction integers \( k, (k+1) \), where \( \mu \leq k \leq (n-2) \), here take the place of \( (n-2), (n-1) \), respectively, there. Hence conclusion (i) holds in case (γ).

Proof of (ii). This result can be reached by the method that was employed in proving Theorem 3 of I, since inequalities analogous to those in (53) of I hold here. Hence Theorem A is true.
7. An Example of Takenouchi. The Takenouchi equation

\[
\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{5}{11}
\]

has, as we presently prove, no solution \( w \) of the type hypothesized in Theorem A. For the Kellogg solution of (6) is \( X = w = (3, 9, 99) \), and it is easily seen that for this \( X \) there does not exist an integer \( \mu \) of the type specified in (2\( ^\circ \)). One may also negate the existence here of the \( w \) of Theorem A by observing with Takenouchi that the set \( (4, 5, 220) \) satisfies (6), while \( (4^{-1} + 5^{-1}) > (3^{-1} + 9^{-1}) \), so that \( (3, 9) = (w_1, w_2) \) does not accord with hypothesis (3\( ^\circ \)).

8. Corollaries of Theorem A. We have seen in I that when \( a = [(c+1)b - 1] \), the \( w \) of Theorem A always exists (case \( \mu = 1 \)). When \( a \neq [(c+1)b - 1] \) there are, as we point out in the corollaries below, numerous cases for which the \( w \) of Theorem A exists with \( \mu = 2 \). In cases where \( \mu > 2 \), we have no better method than that of trial and error for testing a given \( E \)-solution \( X \) to see whether it satisfies hypothesis (3\( ^\circ \)). However, when \( \mu = 2 \) the following Auxiliary Theorem shows that hypothesis (3\( ^\circ \)) is always satisfied when (1\( ^\circ \)) and (2\( ^\circ \)) are. For Theorem A only the case \( r = 1 \), \( \lambda_{r+1} = \lambda_2 = 0 \) of this Auxiliary Theorem is needed; other cases of it are similarly useful in connection with Theorem ia, \( (i = 2, 3, 4, 5) \), of I.

**Auxiliary Theorem.** Let \( X \) be an \( E \)-solution of the equation

\[
\sum_{n,r} \left( \frac{1}{x} \right) + \lambda_{r+1} \sum_{n,r+1} \left( \frac{1}{x} \right) + \cdots + \lambda_s \sum_{n,s} \left( \frac{1}{x} \right) = \frac{b}{a},
\]

where \( X \neq w \), \( w \) being the Kellogg solution of this equation; the \( \lambda_i \), \( (i = r+1, \cdots, s, s > r) \), are integers \( \geq 0 \); \( n \geq r + 2 \); and \( a, b \) are as in Theorem A except that we now suppose that

\[
\frac{1}{w_1 \cdots w_r} - \frac{b w_1 \cdots w_{r-1}}{a w_1 \cdots w_r} 
\]

and that

* See p. 92 of loc. cit. in footnote 3, p. 876, of I.

† This restriction insures the presence of at least two terms free of \( x_n \) in the left member of the equation just displayed. When \( n = r + 1 \), Theorem A has no content beyond the case \( \mu = 1 \), which was handled in I.
Then we conclude* that \((X_1 \cdots (r+1))\) is a set \(\sigma\) in the sense that

\[
\frac{1}{w_1 \cdots w_r} + \frac{1}{w_{r+1}} \left[ \sum_{r} \frac{1}{w} + \frac{\lambda_{r+1}}{w_1 \cdots w_r} \right] = \frac{b}{a} \frac{w_1 \cdots w_{r+1} - 1}{w_1 \cdots w_{r+1}}. 
\]

The proof of this theorem can be made by reasoning of the type that was used on p. 897 of I. Each of the three corollaries below is a case \(\mu = 2\) of Theorem A.

**Corollary 1.** For the equation \(\Sigma_{n,1}(1/x) = 5/17\), with \(n > 2\), the \(w\) of Theorem A is \(w_1 = 4\), \(w_2 = 23\), \(w_{i+1} = 17w_1 \cdots w_i + 1\), \((i = 2, \cdots, n - 2)\), \(w_n = 17w_1 \cdots w_{n-1}\).

**Corollary 2.** For the equation \(\Sigma_{n,1}(1/x) = 3/(3c - 2)\), with \(n > 2\), if \(c \geq 3\) is odd, the \(w\) of Theorem A is defined by the equations \(w_1 = c\), \(w_2 = 2^{-1}(3c^2 - 2c + 1)\), \(w_{i+1} = (3c - 2)w_1 \cdots w_i + 1\), \((i = 2, \cdots, n - 2)\), \(w_n = (3c - 2)w_1 \cdots w_{n-1}\).

**Corollary 3.** For the equation \(\Sigma_{n,1}(1/x) = b/a\), with \(n > 2\) and \(a = [(c+1)b - 2]\), if \(b > 1\) is odd and \(c\) is even, the set \(w\) of Theorem A is defined by \(w_1 = c+1\), \(w_2 = 2^{-1}[a(c+1)+1]\), \(w_{i+1} = aw_1 \cdots w_i + 1\), \((i = 2, \cdots, n - 2)\), \(w_n = aw_1 \cdots w_{n-1}\).

9. Generalizations of Theorem 2, 3 of I.

**Theorem 2a.** Suppose \(a\), \(b\) and \(\mu\), where \(r \leq \mu \leq (n - 1)\), are given positive integers, with \(a\) and \(b\), \(b \leq a\), relatively prime, and that there exists a set of \(n\) numbers \(w = (w_1, \cdots, w_n)\) with the following properties:

(1°) it is an \(E\)-solution of the equation

\[
\Sigma_{n,r}(1/x) = b/a, \quad (1 < r < (n - 1)); 
\]

* If the sign \(<\) in the second relation below were replaced by the sign \(\leq\), \((X_1 \cdots (r+1))\) would still be a set \(\sigma\) in a sense perfectly analogous to that in which the term set \(\sigma\) was used in I.

† The case \(r = 1\) has been treated in Theorem A; the case \(r = (n - 1)\), in I. Hence only the cases \(1 < r < (n - 1)\) remain for consideration.
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\[
(2°) \quad \Sigma_{p,r}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p}
\]

for every positive integral value of \( p \) for which \( \mu \leq p \leq (n-1) \) and for no positive integral value of \( p \) less than \( \mu \);

\( (3°) \) if \( x = X \), where \( X \neq w \), is an \( E \)-solution of the equation in \((1°)\), then for every positive integral value of \( p \) such that \( r \leq p \leq \mu \) for which \( X_1 \cdots p \neq w_1 \cdots p \), the relation \( \Sigma_{p,r}(1/X) \leq \Sigma_{p,r}(1/w) \) holds.

Then \( w_n \) is the largest number that exists in any \( E \)-solution of the equation in \((1°)\) and \( w_n \) appears in no \( E \)-solution of this equation except \( w \).

**Theorem 3a.** If \( x \neq w \) is any \( E \)-solution of the equation considered in Theorem 2a, \( P(x) < P(w) \). [See Theorem 3 of I.]

Since the methods which we have used in proving Theorem A together with those of I suffice to prove the last two theorems we omit their proofs.

The following corollary shows that Theorem ia, \((i = 2, 3)\), has content for a case in which \( \mu > r \).

**Corollary 4.** The Kellogg solution of the equation \( \Sigma_{n,3}(1/x) = 5/11 \), \( n \geq 5 \), is \( x = w \), where \( w_1 = w_2 = 1, w_3 = 3, w_4 = 14, w_{i+1} = 11\Sigma_{i,i-r+1}(w) + 1, (i = 4, \cdots, n-2) \), \( w_n = 11\Sigma_{n-1,n-r}(w) \).

It is easy to show that in this example the \( \mu \) of Theorem 2a exists and equals 4.

10. **Generalizations of Theorems 4, 5.**

**Theorem 4a.** Suppose \( a, b, \mu \) are as in Theorem 2a and that there exists a set of \( n \) numbers \( w > (w_1, \cdots, w_n) \) with the following properties:

\( (1°) \) \( i^t \) is an \( E \)-solution of the equation \( \Sigma_{n,r}(1/x) + \lambda_{r+1}(1/x) = b/a \), in which \( \lambda_{r+1} \) is an integer \( \geq 0 \);

\( (2°) \quad \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w) = \frac{bw_1 \cdots w_p - 1}{aw_1 \cdots w_p} \)

for every positive integral value of \( p \) for which \( \mu \leq p < n \) (and for no positive integral value of \( p \) less than \( \mu \) when \( \mu > r \));

\( (3°) \) if \( x = X \), where \( X \neq w \), is an \( E \)-solution of the equation in \((1°)\), then for every positive integral value of \( p \) such that \( r \leq p \leq \mu \) for which \( X_1 \cdots p \neq w_1 \cdots p \), the relation
\[ \Sigma_{n,r}(1/X) + \lambda_{r+1}\Sigma_{p,r+1}(1/X) \leq \Sigma_{p,r}(1/w) + \lambda_{r+1}\Sigma_{p,r+1}(1/w) \]

holds.

In such a set \( w \), \( w_n \) is the largest number that exists in any \( E \)-solution of the equation in \((1^o)\) and \( w_n \) appears in no \( E \)-solution of this equation except \( w \). Furthermore a similar statement holds when the left member of the equation in \((1^o)\) is replaced by

\[ \Sigma_{n,r}(1/x) + \Sigma_{n,r+1}(1/x) + \cdots + \Sigma_{n,s}(1/x), \]

where, as heretofore, \( s \) is a positive integer and \( r < s \leq n \).

**Theorem 5a.** In each of the two cases of Theorem 4a, if \( X \) is an \( E \)-solution of the given equation and \( \neq w \), the Kellogg solution of that equation, then \( P(X) < P(w) \).

The following corollaries show that the theorems of this section have content in cases where \( \mu = 2 \) when \( r = 1 \).

**Corollary 5.** For the equation \( \Sigma_{n,1}(1/x) + 3\Sigma_{n,2}(1/x) = 5/17 \), with \( n > 2 \), the set \( w \) of Theorem 4a is given by \( w_1 = 4, w_2 = 40, w_{i+1} = 17\left[\Sigma_{i,i}(w) + 3\Sigma_{i,i-1}(w)\right] + 1, \quad (i = 2, \cdots, n-2) \), and \( w_n = 17\left[\Sigma_{n-1,n-1}(w) + 3\Sigma_{n-1,n-2}(w)\right] \).

**Corollary 6.** For the equation \( \Sigma_{n,1}(1/x) + \Sigma_{n,2}(1/x) = 4/13 \), with \( n > 2 \), the \( w \) of Theorem 4a (see last sentence of that theorem) is given by \( w_1 = 4, w_2 = 22, w_{i+1} = 13\left[\Sigma_{i,i}(w) + 3\Sigma_{i,i-1}(w)\right] + 1, \quad (i = 2, \cdots, n-2) \), and \( w_n = 13\left[\Sigma_{n-1,n-1}(w) + 3\Sigma_{n-1,n-2}(w)\right] \).

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ERRATA

The following changes should be made in the present volume (Vol. 40) of this Bulletin:

Page 93, last line of Theorem 2, insert before the words "is that" the words "and that \( f_m(x) \) be continuous."

Pages 413–416, change \( f \) to \( f_0 \) in the following places: in the statement of Theorem 2 on p. 413; in the statement of Theorem 6 on p. 415; and in five places occurring in the first six lines of p. 416.