THE CATEGORY OF THE CLASS Lip \((\alpha, p)\)

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A function \(x(s)\) is said to belong to the class \(\text{Lip}(\alpha, p)\) on the interval \((a, b)\) provided

\[
\left\| x(s + h) - x(s) \right\| \equiv \left( \int_{a}^{b} \left| x(s + h) - x(s) \right|^p ds \right)^{1/p} = O(h^\alpha),
\]

where \(0 < \alpha \leq 1\).

There exist continuous functions which belong to no class \(\text{Lip}(\alpha, p)\). Indeed if \(x(s) \in \text{Lip}(\alpha, p)\), then the Fourier coefficients of \(x(s)\), \(a_n, b_n\), are \(O(n^{-\alpha})\). Now a continuous function may be constructed* such that \(\left| a_{ni} \right| > 1/\log n_i\) for an infinite set of values \(\{n_i\}\). Then for such a function

\[
\frac{\left| a_{ni} \right|}{n_i^{-\alpha}} > \frac{n_i^\alpha}{\log n_i} \neq O(1),
\]

that is, \(a_n \not\in O(n^{-\alpha})\) and hence the continuous function with the Fourier coefficients \(a_n\) belongs to no class \(\text{Lip}(\alpha, p)\).

We prove the following theorem.

**THEOREM.** The subset \(E\) of \(L_p\), \(p \geq 1\), which is \(\sum \text{Lip}(\alpha, p)\) for \(0 < \alpha \leq 1\), is of the first category in \(L_p\).

We employ a method of proof used by S. Banach.† We take the interval \((0, 1)\) as the fundamental interval and assume the functions to be periodic with the period one. Let \(E_{nm}\) be the set of all \(x(s) \in L_p\) such that

\[
\int_{0}^{1} \left| x(s + h) - x(s) \right|^p ds \leq n^p \left| h \right|^{n/m}, \quad (n, m = 1, 2, \ldots).
\]

The sets \(E_{nm}\) are closed. For, let \(x_i(s) \rightarrow x_0(s)\) in \(L_p\). Set

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\[
\begin{align*}
y_t(s) &= x_t(s + h) - x_t(s), \\
y_0(s) &= x_0(s + h) - x_0(s),
\end{align*}
\]
where \( h \) is fixed but arbitrary. Then
\[
\| y_t - y_0 \| \leq \| x_t(s + h) - x_0(s + h) \| + \| x_t(s) - x_0(s) \| = 2\| x_t(s) - x_0(s) \| \to 0.
\]
But \( \| y_t - y_0 \| \to 0 \) implies that \( \| y_t \to \| y_0 \| \), that is,
\[
\int_0^1 | x_0(s + h) - x_0(s) |^p ds \leq n^p | h |^{p/m}.
\]
Moreover \( E \subset \sum_{n, m=1}^{\infty} E_{nm} \). For, if \( x_0(s) \subset E \), then for some value \( \alpha_0 \), \( x_0(s) \subset \text{Lip} (\alpha_0, \rho) \); that is, there exists a number \( M \) such that
\[
\int_0^1 | x_0(s + h) - x_0(s) |^p ds \leq M | h |^{\alpha_0 p}.
\]
To complete the proof we have only to show that every set \( E_{nm} \) is non-dense. Suppose, if possible, that \( E_{nm} \) were not non-dense. Then, since \( E_{NM} \) is closed, it contains a sphere \( K \). Let \( \omega(s) \subset K \subset E_{NM} \) be the center of the sphere and \( r > 0 \) the radius. Let \( g(s) \subset L_p \) be a function of \( E \). Since when \( g(s) \subset E \), \( c \cdot g(s) \subset E \), where \( c \) is a constant not zero, we may assume \( \| g \| < r \). Also
\[
\| g(t + h) - g(t) \| > 2N | h |^{1/M}.
\]
Set \( z(s) = \omega(s) + g(s) \). Then \( z(s) \subset L_p \) and
\[
\| z(t + h) - z(t) \| \geq \| g(t + h) - g(t) \| - \| \omega(t + h) - \omega(t) \| > 2N | h |^{1/M} - N | h |^{1/M} \geq N | h |^{1/M};
\]
that is, \( z(s) \) not \( \subset E_{NM} \). But \( \| z - \omega \| = \| g \| < r \); this means \( z(s) \subset K \subset E_{NM} \), a contradiction.

In exactly the same manner we may prove the following result.

The subset \( EC \) of the space \( C \) of continuous functions is of the first category in \( C \).

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