A THEOREM ON ANALYTIC FUNCTIONS
OF A REAL VARIABLE

BY R. P. BOAS, JR.

1. Introduction. Let \( f(x) \) be a function of class \( C^\infty \) on \( a \leq x \leq b \). At each point \( x \) of \( [a, b] \) we form the formal Taylor series of \( f(x) \),

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (t - x)^k.
\]

This series has a definite radius of convergence, \( \rho(x) \), zero, finite, or infinite, given by \( 1/\rho(x) = \lim_{k \to \infty} |f^{(k)}(x)/k!|^{1/k} \). The function \( f(x) \) is said to be analytic at the point \( x \) if the Taylor development of \( f(x) \) about \( x \) converges to \( f(t) \) over a neighborhood \( |x - t| < c, c > 0 \), of the point; \( f(x) \) is analytic in an interval if it is analytic at every point of the interval.

Pringsheim stated the following theorem.*

**Theorem A.** If there exists a number \( \delta > 0 \) such that \( \rho(x) \geq \delta \) for \( a \leq x \leq b \), \( f(x) \) is analytic in \( [a, b] \).

However, Pringsheim's proof of the theorem is not rigorous. The purpose of this note is to establish this theorem, and, in connection with the proof, a companion theorem of considerable interest in itself.

**Theorem B.** If \( \rho(x) > 0 \) for \( a \leq x \leq b \) (that is, if the Taylor development of \( f(x) \) about each point converges in some neighborhood of the point), the points at which \( f(x) \) is not analytic form at most a nowhere dense closed set.

Theorem B is, in a certain sense, the best possible, since by a theorem of H. Whitney† there exist functions satisfying the


† H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Transactions of this Society, vol. 36 (1934), pp. 63-89. I am indebted to Dr. Whitney for calling my attention to this paper.
conditions of Theorem B and having the points of an arbitrary nowhere dense closed set as singular points.

Theorem B can also be stated in the following equivalent form.

**Theorem B'.** If \( f(x) \) is of class \( C^\infty \) on \([a, b]\) and analytic at no point of \([a, b]\), there must exist an everywhere dense set of points, \( G \), on \([a, b]\) such that the Taylor development of \( f(x) \) about each point of \( G \) is divergent.

We shall need the following lemma.*

**Lemma.** If \( H \) is a perfect point set on the interval \([\alpha, \beta] \), and if \( H = \sum_{n=0}^{\infty} H_n \), where the \( H_n \) are enumerable in number, there exist a value \( n_0 \) of \( n \) and a sub-interval \([\alpha_0, \beta_0] \) such that \( H_{n_0} \) is dense in \( H \cdot [\alpha_0, \beta_0] \).

2. **Proof of Theorem B.** For each \( x \) in \([a, b]\),

\[
\frac{1}{\rho(x)} = \lim_{n \to \infty} \left| \frac{f^{(n)}(x)}{n!} \right|^{1/n} < \infty.
\]

This implies that there exists a finite function \( \mu(x) \) such that

\[
\left| \frac{1}{n!} f^{(n)}(x) \right|^{1/n} \leq \mu(x), \quad (n = 1, 2, \ldots),
\]

or,

\[
| f^{(n)}(x) | \leq n! \mu(x)^n, \quad (n = 1, 2, \ldots).
\]

Let \( E_k \) be the set (not necessarily non-empty) of points \( x \) such that

\[
k \leq \mu(x) < k + 1, \quad (k = 0, 1, 2, \ldots).
\]

It is clear that \([a, b] = \sum_{n=0}^{\infty} E_k \). By the lemma, there is a sub-interval \([\alpha, \beta] \) and an integer \( k_0 \) such that \( E_{k_0} \) is dense in \([\alpha, \beta] \). For every point of \( E_{k_0} \cdot [\alpha, \beta] \),

\[(1) \quad | f^{(n)}(x) | \leq n! \mu(x)^n < n!(k_0 + 1)^n, \quad (n = 1, 2, \ldots).
\]

For every point of \([\alpha, \beta] \cdot C(E_{k_0}) \), (1) holds by continuity. That

is, (1) holds uniformly in \([\alpha, \beta]\).* But this is a well known sufficient condition for \(f(x)\) to be analytic in \([\alpha, \beta]\). The same reasoning applies to any sub-interval of \([a, b] - [\alpha, \beta]\); thus in any sub-interval there is a further sub-interval in which \(f(x)\) is analytic. The points at which \(f(x)\) is not analytic thus form a nowhere dense set, which is obviously closed.

3. Proof of Theorem A. Assume the theorem false; we shall obtain a contradiction. We have, then, a non-empty set \(H\) of points where \(f(x)\) is not analytic, and by Theorem B, \(H\) is closed and nowhere dense.

We first show that \(H\) is perfect. Suppose that \(H\) contained an isolated point \(X\). The function \(f(x)\) is continuous with all derivatives at \(X\); \(f(x)\) is analytic in each of the intervals \(X - h < x < X\) and \(X < x < X + h\), for some \(h > 0\), and can be extended analytically across the point \(X\) in both directions. It follows immediately that \(f(x)\) is analytic at \(X\), so that \(X\) is not a singular point. \(H\) being perfect, from now on we shall confine our attention to an interval \([a, b]\) such that \(b - a < \delta/4\) and \([a, b]\) contains a perfect subset \(E\) of \(H\).

Since by hypothesis
\[
\lim_{n \to \infty} \left| \frac{f^{(n)}(x)}{n!} \right|^{1/n} = \frac{1}{\rho(x)} \leq \frac{1}{\delta}
\]
for every point of \([a, b]\), it follows that for each \(x\) in \(E\) there is an integer \(N_x\) such that
\[
\left| \frac{1}{n!} f^{(n)}(x) \right|^{1/n} < \frac{2}{\delta}, \quad (n \geq N_x);
\]
hence
\[
(2) \quad |f^{(n)}(x)| \leq n! \lambda^n, \quad (\lambda = 2/\delta, \ n \geq N_x).
\]

Let \(E_k\) be the set of points of \(E\) for which \(N_x = k\). By the lemma, there exist a sub-interval \([\alpha, \beta]\) and a value \(k_0\) of \(k\) such that \(E_{k_0}\) is dense in \(E\cdot [\alpha, \beta]\). For \(x\) in \(E_{k_0} \cdot [\alpha, \beta]\), (2) holds for \(n \geq k_0\). For \(x\) in \((E - E_{k_0}) \cdot [\alpha, \beta]\), (2) holds for \(n \geq k_0\), by continuity. Thus (2) holds uniformly for \(x\) in \(E\cdot [\alpha, \beta]\), \(n \geq k_0\).

* This fact can be obtained as a special case of a general theorem in the theory of operations, which is established by similar reasoning; see S. Banach, op. cit., p. 19 (Theorem 11).
Let \((x_0, y_0)\) be a complementary interval of the nowhere dense set \(E \cdot [\alpha, \beta]\). Then the Taylor series
\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k
\]
converges to \(f(x)\) for \(x_0 \leq x < y_0\). This follows at once from the facts that \(f^{(k)}(x)\) is continuous on \(x_0 \leq x < y_0\) for \(k = 0, 1, 2, \ldots\), and that \(y_0 - x_0 < \delta/4 < \delta\).

Define an auxiliary function \(\phi(x) = (\beta_1 - \alpha)/(\beta_1 - x)\), where \(\delta/4 > \beta_1 - \beta > \beta - \alpha > 0\). The function \(\phi(x)\) is analytic on \([\alpha, \beta]\) and is represented over the whole of \([\alpha, \beta]\) by its Taylor development about any point of \([\alpha, \beta]\). We have
\[
\phi^{(k)}(x) = \frac{(\beta_1 - \alpha) \cdot k!}{(\beta_1 - x)^{k+1}} \geq \frac{(\beta_1 - \alpha) \cdot k!}{(\beta_1 - \alpha)^{k+1}} \geq k!\lambda^k,
\]
\((k = 0, 1, 2, \ldots; \alpha \leq x \leq \beta)\).

Now form \(\psi(x) = \phi(x) - f(x)\). The function \(\psi(x)\) is represented by its Taylor development about \(x_0\) for \(x_0 \leq x < y_0\); for \(n \geq k_0\), \(\psi^{(n)}(x_0) \geq 0\) and \(\psi^{(n)}(y_0) \geq 0\) by (2). Hence for \(n \geq k_0\), \(\psi^{(n)}(x) \geq 0\) for \(x_0 \leq x \leq y_0\), since we may differentiate a power series term-wise any number of times in the interior of its interval of convergence, so that \(\psi^{(n)}(x)\) is represented over \(x_0 \leq x < y_0\) by a series of non-negative terms, for \(n \geq k_0\). This reasoning applies to any complementary interval of \(E \cdot [\alpha, \beta]\), with the same function \(\psi(x)\). Hence \(\psi^{(n)}(x) \geq 0\) for \(\alpha \leq x \leq \beta\), \(n \geq k_0\). By a well known theorem of S. Bernstein, \(\psi^{(k_0)}(x)\) is analytic for \(\alpha \leq x \leq \beta\), and consequently \(\psi(x)\) is analytic in the same interval. But then \(f(x) = \phi(x) - \psi(x)\) is analytic in \([\alpha, \beta]\), contrary to the hypothesis that \(H\) was not an empty set. Hence \(H\) is an empty set, and the theorem is proved.

HARVARD UNIVERSITY