ON A CERTAIN NON-LINEAR ONE-PARAMETER SYSTEM OF HYPERSURFACES OF ORDER $n$ IN $r$-SPACE

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Consider a linear $\infty$-system, where

$$\rho \leq \frac{(n+r)!}{n!r!} - 1,$$

of hypersurfaces of order $n$, which may have $\sigma$ base points, in an $r$-space, $S$. Denote this system by $|W|$.

Now let $\nu_1 + \nu_2 + \cdots + \nu_t$ projectively related curves $C_{11}, C_{12}, \cdots, C_{1\nu_1}, C_{21}, C_{22}, \cdots, C_{2\nu_2}, \cdots, C_{t\nu_t}$ of orders

$$n_{11}, n_{12}, \cdots, n_{1\nu_1}, n_{21}, n_{22}, \cdots, n_{2\nu_2}, \cdots, n_{t\nu_t},$$

respectively, and all of genus $\rho$, be given in the same $r$-space $S_r$. To a point on any one of the curves corresponds a definite point on each of the other curves. We assume that none of the given curves passes through any of the $\sigma$ base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. Let $P_{11}, P_{12}, \cdots, P_{t\nu_t}$ be a set of corresponding points, the point $P_{i\nu_i}$ being on the curve $C_{ij_i}$, $(i = 1, 2, \cdots, t; j_i = 1, 2, \cdots, \nu_i)$. If

$$\nu_1 + 2\nu_2 + \cdots + t\nu_t = \rho \leq \frac{(n + r)!}{n!r!} - 1,$$

there is one and only one hypersurface of the system $|W|$ such that $1, 2, \cdots, t$ of the points of its intersection with each of the $\nu_1$ curves $C_{1j_1}$, $\nu_2$ curves $C_{2j_2}, \cdots, \nu_t$ curves $C_{tj_t}$, will coincide with $P_{1j_1}, P_{2j_2}, \cdots, P_{tj_t}$, respectively. Denote such a hypersurface by $V_{r-1}^n$. As the corresponding points describe their respective curves, $V_{r-1}^n$ describes a non-linear one-parameter system, $\{V\}$, of hypersurfaces of order $n$ in $S_r$. In this paper we propose to determine the number, $N_0$, the order of the system, of the hypersurfaces of the system passing through a given
point and also the number, $N_k$, of those tangent to a given $k$-space for $k = 1, 2, \cdots, r$. The symbol $N_r$ means the number of the hypersurfaces that have each a node.

In the following, we shall give two determinations of the number $N_0$: the one by the use of the theory of correspondence and the other by the aid of the following known proposition.*

(A) Let there be given $q$ varieties $V_{x_1}^{m_1}, V_{x_2}^{m_2}, \cdots, V_{x_q}^{m_q}$ of orders $m_1, m_2, \cdots, m_q$, respectively, such that $V_{x_1}^{m_1}$ is the locus of $\infty^1 (x_1-1)$-spaces. If there exists a one-to-one correspondence between the elements of these varieties, then the locus of the $\infty^1 (x_1 + x_2 + \cdots + x_q - 1)$-spaces determined by corresponding elements is a $V_{x_1 + x_2 + \cdots + x_q}$ of order $m_1 + m_2 + \cdots + m_q$.

We now determine $N_0$ by the theory of correspondence. We commence with the case $\nu_1 = \rho = 2, \nu_2 = \nu_3 = \cdots = \nu_t = 0$. The system $\{V\}$ now consists of those hypersurfaces of the net $|W|$ which pass through pairs of corresponding points on the two given curves $C_{11}, C_{12}$. The desired number is the number of hypersurfaces of $\{V\}$ passing through a given point, say $A$. Let us make a hypersurface $W_{r-1}^a$ of $|W|$ pass through $A$ and a point $P_{11}$ of $C_{11}$. This $W_{r-1}^a$ meets $C_{12}$ in $n_{12}$ points $Q_{12}, Q_{12}', \cdots$. If one of these points happens to coincide with the point $P_{12}$ corresponding to $P_{11}$ on $C_{11}$, then $W_{r-1}^a$ is a $V_{nr}$ of $\{V\}$. In general, this does not happen. Now pass another hypersurface $W_1^a$ of $|W|$ through $A$ and one of the points $Q_{12}, Q_{12}', \cdots$, say $Q_{12}$. This $W_1^a$ meets $C_{11}$ in $nn_{11}$ points $P_{11}, P_{11}', \cdots$, to which correspond $nn_{11}$ points $P_{12}, P_{12}', \cdots$ on $C_{12}$. We see that we have thus established a correspondence on the curve $C_{11}$ such that to each of the points $Q_{12}, Q_{12}', \cdots$ correspond $nn_{11}$ points $P_{11}, P_{11}', \cdots$ and to each of the latter correspond $nn_{12}$ points of the former. If a united point occurs, then the two hypersurfaces $W_{r-1}^a, W_1^a$ become coincident with a $V_{nr}$ of $|W|$. The correspondence being obviously of valence zero, the number of united points, and therefore the order of $\{V\}$, is $n(n_{11} + n_{12})$.

Suppose now $\nu_1 = \rho = 3, \nu_2 = \nu_3 = \cdots = \nu_t = 0$. Choose a $W_{r-1}^a$ of the net $|W|$ that passes through a given point $A$ and a pair

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* A full discussion of this proposition is found in Edge, *On the quartic developable*, Proceedings of the London Mathematical Society, (2), vol. 33, pp. 52-65. The statement above is quoted verbatim from B. C. Wong, *On the number of stationary tangent $S_{r-1}$'s to a $V_k^n$ in an $S_{1k+k-1}$*, this Bulletin, vol. 39 (1933), pp. 608-610.
of corresponding points \( P_{11}, P_{12} \) on \( C_{11}, C_{12} \). This \( W_{r-1} \) meets the third given curve \( C_{13} \) in \( nn_{13} \) points \( Q_{13}, Q_{13}', \ldots \), none of which, in general, coincides with the point \( P_{13} \) corresponding to \( P_{11} \) and \( P_{12} \). Through each of these points \( Q_{13}, Q_{13}' , \ldots \) there are, according to the result just found, \( n(n_{11} + n_{12}) \) hypersurfaces each containing \( A \) and a pair of points \( P_{11}, P_{12} \). Then on \( C_{13} \) there are \( n(n_{11} + n_{12}) \) points \( P_{13}, P_{13}', \ldots \) corresponding to as many pairs on \( C_{11}, C_{12} \). Now we have on \( C_{13} \) an \( (n_{11}n + n_{12}n_{13}n) \)-correspondence of valence zero between the points \( P_{13}, P_{13}', \ldots \) and the points \( Q_{13}, Q_{13}' , \ldots \). The number of united points, and therefore the number of the hypersurfaces of \( \left| W \right| \) passing through \( A \) and a set of corresponding points on \( C_{11}, C_{12}, C_{13} \), is \( n(n_{11} + n_{12} + n_{13}) \).

If we continue reasoning in this manner, we shall find that, for the case \( v_1 = \rho, v_2 = v_3 = \cdots = v_t = 0 \), the order of \( \{ V \} \) is \( n(n_{11} + n_{12} + \cdots + n_{1v}) \) or \( n \sum_{j=1}^{v} n_{1j} \).

Now suppose \( v_1 = \rho - 2, v_2 = 1 \). Then the system \( \{ V \} \) consists of all those hypersurfaces of the \( \infty \)-system \( \left| W \right| \) which contain a set of corresponding points \( P_{1j}, (j = 1, 2, \cdots, v_1) \), on the \( v_1 \) curves \( C_{1j} \) and have a contact with \( C_{21} \) at the point \( P_{21} \) corresponding to \( P_{1j} \). Select a \( W_{r-1} \) of \( \left| W \right| \) passing through a given point \( A \) and a fixed set of points \( P_{1j} \) and having a point of contact with \( C_{21} \). Since a pencil of hypersurfaces of order \( n \) contains \( 2(n_{21}n - 1 + p) \) members tangent to a given curve of order \( n_{21} \) and genus \( p \), there are \( 2(n_{21}n - 1 + p) \) such hypersurfaces and hence there are as many points of contact \( T_{21}, T_{21}', \cdots \) on \( C_{21} \). None of these, in general, coincides with \( P_{21} \). Now pass a hypersurface \( W_{r-1}' \) of \( \left| W \right| \) through the points \( P_{1j} \) on the curves \( C_{1j} \), tangent to \( C_{21} \) at one of the points \( T_{21}, T_{21}', \cdots \), say \( T_{21} \). There are, according to the result of the preceding paragraph, \( n \sum_{j=1}^{v} n_{1j} \) such hypersurfaces giving rise to as many sets of corresponding points \( P_{1j}, P_{1j}', \cdots \), to which correspond as many points \( P_{21}, P_{21}', \cdots \), on \( C_{21} \). Thus, we have established an \( [n \sum_{j=1}^{v} n_{1j}, 2(n_{21}n - 1 + p)] \)-correspondence also of valence zero between the points \( P_{21}, P_{21}', \cdots \), and the points \( T_{21}, T_{21}', \cdots \), on \( C_{21} \). The number of united points in this correspondence which gives the order of \( \{ V \} \) is therefore \( n \sum_{j=1}^{v} n_{1j} + 2(n_{21}n - 1 + p) \).

These particular cases are sufficient to indicate the method used. Reasoning in exactly the same manner for all the differ-
ent values of the $\nu$'s satisfying (I), we find the general result

$$N_0 = \sum_{i_1=1}^{r_1} n_{i_1} n + 2 \sum_{i_2=1}^{r_2} (n_{2i} n - 1 + \rho)$$

$$+ 3 \sum_{i_3=1}^{r_3} (n_{3i} n - 2 + 2\rho) + \cdots + t \sum_{i_t=1}^{r_t} [n_{ti} n + (t - 1)(\rho - 1)],$$

or

$$N_0 = \sum_{i=1}^{t} i \sum_{i_i=1}^{n_i} [n_{i_i} n + (i - 1)(\rho - 1)],$$

where $i [n_{i_i} n + (i - 1)(\rho + 1)]$ is the number of hypersurfaces of order $n$ of an $\infty^{i-1}$-system of hypersurfaces such that $i$ of the points of intersection of each of them with a given curve of order $n_{i_i}$ are coincident.

Now we determine $N_0$ by the aid of (A). Let the hypersurfaces of $\{W\}$, which may have $\sigma$ base points, represent upon $S_r$ an $r$-dimensional variety $\Phi_{r,n-\sigma}$ of order $n'-\sigma$ in a $\rho$-space $S_\rho$. The $\nu_1 + \nu_2 + \cdots + \nu_t$ given curves, none of which is supposed to pass through any of the $\sigma$ base points, are the images of curves $\Gamma_{i_i}$ of order $n_{i_i} n$ on $\Phi_{r,n-\sigma}$ whose points are also in a one-to-one correspondence. Let $R_{1i_1}, R_{2i_2}, \cdots$ be a set of corresponding points, the point $R_{i_i}$ being on the curve $\Gamma_{i_i}$. Corresponding to a hypersurface $V_{r-1}$ of the system $\{V\}$ is a section $\Theta_{r-1}^{n-\sigma}$ of $\Phi_{r,n-\sigma}$ by a $(\rho - 1)$-space which contains a set of points $R_{1i_1}$ on the curves $\Gamma_{i_1}$, a set of tangent lines at the points $R_{2i_2}$ on the curves $\Gamma_{3i_3}$, a set of osculating planes at the points $R_{3i_3}$ on the curves $\Gamma_{4i_4}$, $\cdots$. The $\infty^1 (\rho - 1)$-spaces of the nature just described form an $\infty^1$-system to which corresponds our system $\{V\}$ of hypersurfaces. By applying (A) we find that the order of the system of $(\rho - 1)$-spaces is, since the $i$-dimensional developable of the curve $\Gamma_{i_i}$ is of order $i [n_{i_i} n + (i - 1)(\rho - 1)]$, the same as (1). Now through a given point $A'$ which may be, without loss of generality, placed upon $\Phi_{r,n-\sigma}$, pass the same number of $(\rho - 1)$-spaces of the system and each such $(\rho - 1)$-space intersects $\Phi_{r,n-\sigma}$ in a $\Theta_{r-1}^{n-\sigma}$ passing through $A'$ to which corresponds a $V_{r-1}$ of $\{V\}$ passing through a given point $A$, the image of $A'$. Thus, the determination is complete.

Hitherto we have assumed that none of the given curves
passes through any of the base points of $|W|$ and that none of the intersections, if there be any, of any two of the curves is a self-corresponding point. If, however, a curve $C_{ii}$ passes through one of the base points, we must deduct $i$, and if any two whatever of the curves intersect in a self-corresponding point, we must deduct unity from the general value of $N_0$ which we have just derived.

As an example consider a linear $\infty^6$-system $|K|$ of quartic curves in a plane $\phi$ with 8 base points. Let three projectively related cubic curves $\gamma^3, \gamma'^3, \gamma''^3$ of genus unity be given in the plane, none of the intersections of the curves being a self-corresponding point. Select a quartic of $|K|$ such that one of its intersections with $\gamma^3$ coincides at $P$, two of its intersections with $\gamma'^3$ coincide at $P'$, and three of its intersections with $\gamma''^3$ coincide at $P''$, where $P, P', P''$ are a set of corresponding points. There are $\infty^1$ such quartic curves forming a non-linear pencil, \{C\}. Now the quartics of $|K|$ represent upon $\phi$ a surface $\Phi^8$ of order 8 in $S_8$ upon which lie three projectively related curves $\Gamma^{12}, \Gamma''^{12}, \Gamma'''^{12}$ all of order 12 and genus 1, of which $\gamma^3, \gamma'^3, \gamma''^3$ are the images in $\phi$. The locus of tangent lines of $\Gamma^{12}$ is of order 24 and the locus of osculating planes of $\Gamma''^{12}$ is of order 36. Let $R, R', R''$ be a set of corresponding points on the curves. Then we say that the tangent $t'$ to $\Gamma^{12}$ at $R'$ and the osculating plane $\pi''$ to $\Gamma''^{12}$ at $R''$ correspond to the point $R$ on $\Gamma^{12}$. The 5-spaces determined by $R, t', \pi''$ will describe an $\infty^1$-system of 5-spaces such that $N_0 = 72$ of them pass through a given point $A'$ which may be placed on $\Phi^8$. Therefore the system \{C\} of quartic curves contains 72 members passing through a given point $A$ of $\phi$.

Suppose the curve $\gamma^3$ passes through a base point of $|K|$. Then the corresponding curve $\Gamma^{12}$ on $\Phi^8$ is composed of a $\Gamma^{11}$ and a line. Discarding the line or deducting unity, we have $N_0 = 71$. If $\gamma'^3$ alone contains a base point, the corresponding curve $\Gamma''^{12}$ degenerates into a line, to be disregarded, and a $\Gamma''^{11}$ whose developable surface is of order 22. Therefore, we deduct 2 and now $N_0 = 70$. Finally, let $\gamma''^3$ alone go through a base point. The curve $\Gamma'''^{12}$ on $\Phi^8$ is made up of a line, also to be disregarded, and a $\Gamma'''^{11}$ the locus of whose osculating planes is of order 33. Deducting 3, we now have $N_0 = 69$.

Now let one of the intersections of $\gamma^3, \gamma'^3$ be a self-corresponding-
ing point. Then $\Gamma^{12}$, $\Gamma^{13}$ also have a self-corresponding point $R=R'$ to which corresponds the point $R''$ on $\Gamma^{12}$. There is a linear pencil of 5-spaces passing through the tangent line $t'$ at the self-corresponding point $R=R'$ and containing the osculating plane of $\Gamma^{12}$ at $R''$. Disregarding this pencil, we deduct 1 and the result is $N_0=71$. If $\gamma^{13}$, $\gamma^{12}$ have a self-corresponding point in common, then one of the intersections of $\Gamma^{12}$, $\Gamma^{13}$ is a self-corresponding point $R'=R''$ to which corresponds a point $R$ of $\Gamma^{12}$. There is a linear pencil of 5-spaces determined by $R$ and the tangent line to $\Gamma^{12}$ at $R'$ and the osculating plane of $\Gamma^{12}$ at $R''\equiv R'$. Deducing 1, we have $N_0=71$.

We shall next proceed to determine $N_k$, the number of the hypersurfaces of $\{V\}$ tangent to a given $k$-space in $S_r$. We find it convenient to use the following method. We set up a one-to-one correspondence between the points of a $p$-space $S_p$ and the hypersurfaces of the $\infty^p$-system $|W|$. Corresponding to the $\infty^{p-1}$ hypersurfaces of $|W|$ that pass through a given point $A$ are the $\infty^{p-1}$ points of a $(p-1)$-space $S_{p-1}$ of $S_p$, and corresponding to the hypersurfaces of $\{V\}$ are the points of a curve $\Delta$. Since there are given by (1), as we have seen, $N_0$ hypersurfaces of $|W|$ passing through $A$ and belonging to $\{V\}$, there must be $N_0$ points of $S_p$ common to $S_{p-1}$ and $\Delta$. Hence $\Delta$ is of order $N_0$.

Let a $k$-space $S_k$ be given in $S_r$. Contact being one condition, there are $\infty^{p-1}$ hypersurfaces of $|W|$ tangent to $S_k$, and to these contact hypersurfaces correspond $\infty^{p-1}$ points of a locus $\Sigma_{p-1}^M$ in $S_p$. By the methods of analytic geometry we find without difficulty that the order $M$ of $\Sigma_{p-1}^M$ is $M=(k+1)(n-1)^k$. All those hypersurfaces of $|W|$ belonging to $\{V\}$ and tangent to $S_k$ are given by all those points of $S_p$ common to $\Delta^{N_0}$ and $\Sigma_{p-1}^M$. Therefore, the number of hypersurfaces of $\{V\}$ tangent to $S_k$ is the number of the points in which $\Delta^{N_k}$ intersects $\Sigma_{p-1}^M$ and is therefore equal to $N_k=MN_0=(k+1)(n-1)^kN_0$.

For $k=1,2$, then, $N_1=2(n-1)N_0$ and $N_2=3(n-1)^2N_0$ are, respectively, the number of hypersurfaces of $\{V\}$ tangent to a given line and the number of those tangent to a given plane. If $k=r$, we have $N_r=(r+1)(n-1)^rN_0$ members of the system that have each a node.

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