A CONNECTEDNESS THEOREM IN ABSTRACT SETS*

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This note gives a variation of a theorem of Sierpinski and Saks.† The theorem is valid in spaces which have the Borel-Lebesgue property (Axiom I of Saks‡) and which satisfy axioms (A), (B), (C), and (6) as given by Hausdorff.§ We use the term connected for a closed set to mean that the set cannot be expressed as the sum of two mutually exclusive non-vacuous, closed sets.||

THEOREM. Let \( F \) be a collection of closed sets at least one of which is compact. Let \( F \) contain more than one element and let it be true that the sets of each finite sub-collection of \( F \) have a non-vacuous, connected set in common when this sub-collection contains at least two elements of \( F \). Under these hypotheses, there is a closed, non-vacuous, connected set common to all of the sets of collection \( F \).

PROOF. Let \( F_0 \) be a compact member of collection \( F \) and let \( K \) be the set of points common to all of the sets of collection \( F \).

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* Presented to the Society, December 1, 1934.
‡ Saks, ibid., p. 2.
§ Mengenlehre, 1927, pp. 228–229.
|| The notion of limit point may be defined and this definition used to describe connectedness. We use domain and open set interchangeably.
Saks* shows that $K$ is non-vacuous while the closure of $K$ is an immediate consequence of the closure of the sets of collection $F$ (since any point of the complement of $K$ has a neighborhood which belongs to the complement of some one of the sets of collection $F$ and hence belongs to the complement of $K$). It remains to show that $K$ is connected. Suppose $K = K_1 + K_2$, where $K_1$ and $K_2$ are mutually exclusive, non-vacuous, closed sets. The collection $C$ composed of the complements of the sets of collection $F$ is a set of domains that covers $F_0 - K$. By axiom (6), for each point $p$ of $K_1$ there exist mutually exclusive domains $G_{1p}$ and $G_{2p}$ such that $p \in G_{1p}$, $K_2 \subseteq G_{2p}$. Let $[G_{1p}]$ and $[G_{2p}]$ be the collections of domains obtained in this manner for all points of $K_1$. The set $K_1$ is closed and compact and hence has the Borel-Lebesgue property. Let $G_1, \ldots, G_n$ be a finite sub-collection of $[G_{1p}]$ which covers $K_1$ and let $H_1, \ldots, H_n$ be the corresponding members of $[G_{2p}]$. If $H$ denotes the common part of $H_1, \ldots, H_n$, then $H$ is a domain that covers $K_2$ (this follows from a theorem stated by Hausdorff, loc. cit., page 229, line 4) while $G = G_1 + G_2 + \ldots + G_n$ is a domain that covers $K_1$. Furthermore, $H$ and $G$ have no point in common since $G_i$ and $H_i$ are mutually exclusive sets. The collection $C$ together with $G$ and $H$ cover the closed and compact set $F_0$. The Borel-Lebesgue property yields a finite collection $C_1, C_2, \ldots, C_m, G, H$, of these sets that covers $F_0$ while the hypotheses of the theorem together with the method of construction of the covering sets force this collection to contain $G, H$, and at least one of the sets $C_i$. Let $F_i$ be the complement of $C_i$ and let $Q$ be the set common to $F_0, F_1, \ldots, F_m$. The set $Q$ contains $K_1$ and $K_2$ and is covered by $G$ and $H$ (since $Q$ belongs to $F_0$ and the complements of $C_1, C_2, \ldots, C_m$). Since $G$ and $H$ are mutually exclusive, it follows that $Q$ is not connected. This contradicts the hypothesis that any finite collection of two or more of the sets of $F$ has a connected set in common and yields the theorem.